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Prediction of the buckling stress of intermediate wooden columns using the secant modulus

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Abstract We made several buckling tests of wooden columns with intermediate slenderness ratios (λ) and examined the empirical formulas. On the basis of the examination, we formulated an equation for predicting the buckling stress (σ) of an intermediate wooden column. Sitka spruce (*Picea sitchensis* Carr.) and buna (Japanese beech, *Fagus crenata* Bl.) were used for the studies. A compressive load was applied on the specimen supported with pin ends, and the buckling stress was predicted by the tangent modulus theory and two empirical equations: those of Tetmajer and Newlin-Gahagan. The predicted σ–λ relations were compared with the test results, and the applicability of these predictions were examined. Based on the comparisons, we formulated an equation that can predict the σ–λ relations of materials with various stress–strain characters in the plastic strain range.

Key words Buckling stress · Intermediate column · Tangent modulus theory · Empirical formulas · Secant modulus

Introduction

In a previous paper, we examined the applicability of Engesser's tangent modulus theory to predicting the buckling stress of a wooden column with an intermediate slenderness ratio, and we suggested that the buckling stress predicted by this theory is seriously influenced by the formulation of the stress–strain relation that gives the tangent modulus.[1] The procedure for predicting buckling stress by the tangent modulus theory is rather complicated. Hence other equations experimentally obtained are usually used to predict the buckling stress of an intermediate wooden column. Among these equations, the Tetmajer[2] and Newlin-Gahagan[3] equations are famous because of their simplicity. They do, however, neglect the stress–strain relation that influences the value of buckling stress, and we thought that these equations were not sufficient. In this study we examined these empirical equations and propose a new equation for predicting the buckling stress of an intermediate wooden column.

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Theories

When a column is long enough to buckle under the elastic stress condition, the buckling stress can be represented by Euler's formula as[4]

$$\sigma = \frac{\pi^2 EI}{l^2 A} = \frac{\pi^2 E}{\lambda^2} \tag{1}$$

where E, l, A, I, and λ are Young's modulus, column length, cross-sectional area, moment of inertia of the column cross section, and slenderness ratio, respectively.

From Eq. (1), the buckling stress coincides with the yield stress (Y) at the slenderness ratio of λ_y represented as

$$\lambda_y = \pi\sqrt{\frac{E}{Y}} \tag{2}$$

When the slenderness ratio is smaller than λ_y, buckling occurs in the plastic region. To predict buckling stress in the plastic region, the tangent modulus theory was proposed by Engesser.[4] For the intermediate column, Young's modulus is replaced by the tangent modulus (E_t), which is the local slope of the stress–strain diagram in the plastic strain range. Hence the buckling stress is given as

$$\sigma = \frac{\pi^2 E_t}{\lambda^2} \tag{3}$$

The tangent modulus E_t is written as

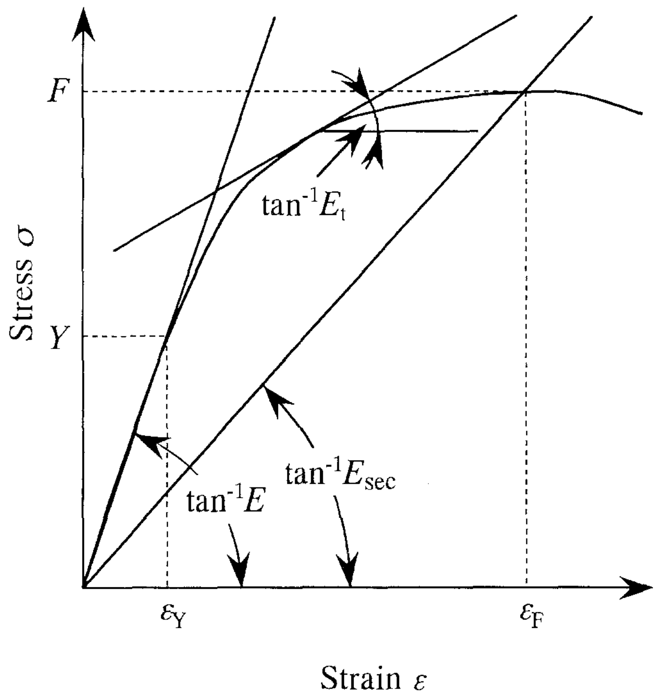

Fig. 1. Stress–strain diagram. Notes: E_t and E_{sec} represent the tangent modulus and the secant modulus, respectively; Y and F represent the yield and crushing stresses, respectively; and ε_Y and ε_F represent the strains at the occurrence of yielding and crushing, respectively

$$E_t = \frac{d\sigma}{d\varepsilon} \tag{4}$$

and is graphically represented as in Fig. 1. In a previous paper we examined the stress–strain formulas used for predicting the tangent modulus and recommended the Ylinen-type logarithmic function represented as[1,5]

$$\varepsilon = \frac{\sigma}{E} + \alpha(F-Y)\ln\left(\frac{F-Y}{F-\sigma}\right) - \alpha(\sigma - Y) \tag{5}$$

where F is the upper boundary of a stress–strain relation (crushing stress), and α is the material parameter that represents the volume of the plastic strain range. When the value of α is large, the stress–strain relation has a large plastic strain range. Differentiating these formulas with respect to σ, the tangent moduli (E_t) are given by

$$E_t = \frac{E}{1 + \alpha\frac{\sigma - Y}{F - \sigma}E} \tag{6}$$

Substituting Eq. (6) into Eq. (3), the buckling stress (σ) satisfies the following quadratic equation:

$$P\sigma^2 + Q\sigma + R = 0 \tag{7}$$

where

$$P = 1 - \alpha E$$
$$Q = \alpha EY - F - \frac{\pi^2 E}{\lambda^2} \tag{8}$$
$$R = \frac{\pi^2 EF}{\lambda^2}$$

Because $\sigma > 0$, the solution of Eq. (7) is derived as

$$\sigma = \frac{-Q + \sqrt{Q^2 - 4PR}}{2P} \tag{9}$$

As mentioned above, the buckling stress predicted by the tangent modulus theory is seriously influenced by the stress–strain formula giving the tangent modulus; additionally, the tangent modulus theory is inconvenient because the overall stress–strain relation should be given. Thus the other formulas empirically obtained had been often used for the prediction of buckling stress of an intermediate wooden column. One of the formulas was proposed by Tetmajer.[2] He formulated the buckling stress/slenderness ratio relation by a quadratic formula as

$$\sigma = F\left(1 - k_1\lambda + k_2\lambda^2\right) \tag{10}$$

where k_1 and k_2 are the material parameters. Later his equation was simplified to a linear function as follows.[2]

$$\sigma = F\left(1 - k_1\lambda\right) \tag{11}$$

When the σ–λ relation represented by Eq. (10) connects with that given by Euler's equation at the yield stress Y, k_1 satisfies the following equation.

$$k_1 = \frac{F - Y}{F\lambda_y}, \tag{12}$$

Another empirical formula was proposed by Newlin and Gahagan.[3] They formulated the buckling stress by a power function of the slenderness ratio with the coordinates of $(0, F)$ and (λ_y, Y) as follows.

$$\sigma = F\left\{1 - \left(1 - \frac{Y}{F}\right)\cdot\left(\frac{\lambda}{\lambda_y}\right)^{\frac{2Y}{F-Y}}\right\} \tag{13}$$

Experiment

Specimens

Small clear specimens of Sitka spruce (*Picea sitchensis* Carr.) and buna (Japanese beech, *Fagus crenata* Bl.) were used. The density of spruce was 0.48 and that of buna 0.60. These specimens were conditioned at 20°C and 65% relative humidity (RH) before and during the tests; therefore the moisture content of specimen was kept constant at 12%.

Buckling test

Column specimens were cut with rectangular sections of 30mm (radial direction) × 20mm (tangential direction). The lengths of the specimens varied: 70, 100, 160, 220, 280, 340, 400, 460, and 700mm. Strain gauges (gauge length 2mm) (Tokyo Sokki, Tokyo, Japan) were bonded on both the longitudinal-radial (LR) planes for the measurement of normal strains in the loading direction.

Table 1. Material parameters of each species

Species	E (kgf/cm^2)	Y (kgf/cm^2)	F (kgf/cm^2)	α	E_{sec} (kgf/cm^2)
Spruce	166000	260	450	1.00×10^{-6}	155000
Buna	102000	270	490	1.51×10^{-5}	48500

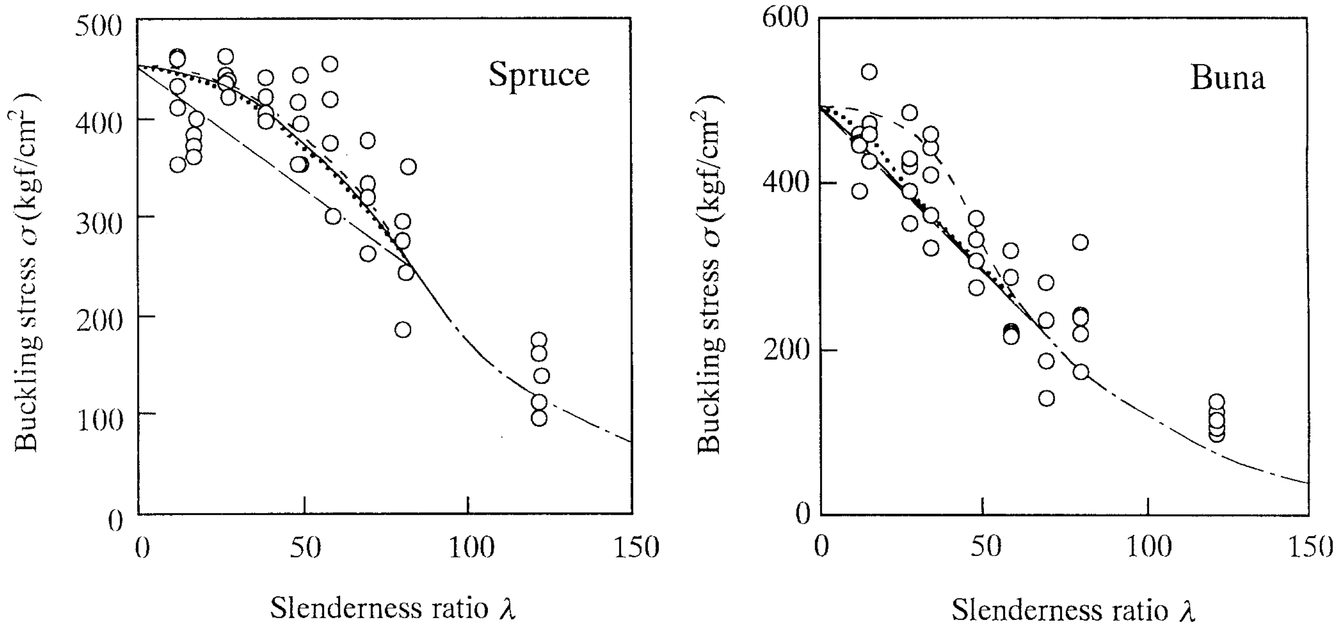

Fig. 2. Relation between the buckling stress (σ) and the slenderness ratio (λ). *Circles*, experimental data; *long-short dashes*, Euler's formula; *dotted lines*, tangent modulus theory; *long dashes*, Tetmajer's formula; *short dashes*, Newlin-Gahagan's formula; *solid line*, our proposal

The column was supported with pin ends, and the load was applied with a velocity of 1 mm/min. The buckling load was determined at the occurrence of strain bifurcation.

The buckling stress was independently predicted by the Tetmajer equation [Eq. (11)] and the Newlin-Gahagan equation [Eq. (13)], and the predictions were compared with each other. Five specimens were used for one test condition.

Compression tests of short column specimens

To obtain the stress–strain relations, compression tests of short column specimens with the dimensions 40 mm (L) × 20 mm (R) × 20 mm (tangential, or T) were made. The obtained stress–strain relations were regressed to Eq. (5), and the material parameters E, Y, F, and α were obtained. By substituting these parameters into Eq. (9), the buckling stress σ corresponding to the slenderness ratio λ was predicted by the tangent modulus theory.

Results and discussion

Table 1 shows the material parameters, E, Y, F, and α for each species. The values of α indicate that the spruce had a small plastic strain range in the stress–strain relation, whereas the buna had a large plastic strain range.

With these parameters, the buckling stresses of the specimens with intermediate slenderness ratio were predicted by the tangent modulus theory as in Fig. 2. The experimental results show that the relation between the buckling stress σ and slenderness ratio λ of spruce is convex in the intermediate column range, whereas that of buna is rather linear. These tendencies are predicted properly by the tangent modulus theory when the stress–strain relation is regressed to the logarithmic function. Of course, the buckling stress predicted by this theory is seriously influenced by the formulation of the stress–strain relation that gives the tangent modulus, and the procedure for predicting the buckling stress by tangent modulus theory is complicated.[1]

Figure 2 also shows the σ–λ relations predicated by the Tetmajer and Newlin-Gahagan equations. We evaluated the applicability of these equations to the experimental data by summing the squares of the differences between the experimental and predicted buckling stresses. This value is represented as

$$c = \Sigma\left(f(\lambda_m) - \sigma_m\right)^2 \quad (14)$$

where σ_m is the buckling stress at the slenderness ratio of λ_m experimentally obtained, and $f(\lambda_m)$ is the buckling stress at the same slenderness ratio predicted by an equation, which in this paper is the Tetmajer or Newlin-Gahagan equation. When the value of c is large, the equation deviates from the experimental data. Thus Tetmajer's linear function was applicable for predicting the buckling stress of buna, whereas the Newlin-Gahagan's power function was applicable for spruce.

The Tetmajer and Newlin-Gahagan equations have no parameters that can affect the stress–strain characteristics; hence their equations are not convenient for predicting the buckling stresses of material with various stress–strain characteristics in the plastic strain range. To reduce this drawback, we proposed an equation with a parameter representing the stress–strain relation in the plastic strain range. Similar to the Newlin-Gahagan equation, we formulated the σ–λ relationship by a power function with the coordinates $(0, F)$ and (λ_y, Y) as follows:

$$\sigma = F\left\{1 - \left(1 - \frac{Y}{F}\right) \cdot \left(\frac{\lambda}{\lambda_y}\right)^{\frac{2E_{sec}}{E}}\right\} \quad (15)$$

where E_{sec} is the secant modulus at $\sigma = F$ (Fig. 1) and is defined by the failure stress and the failure strain ε_F as

$$E_{sec} = \frac{F}{\varepsilon_F} \quad (16)$$

The power term in our proposal $(2E_{sec}/E)$ affects the stress–strain relation in the plastic region. When the plastic strain range is larger the value of E_{sec}/E decreases, and so the σ–λ relation is close to the Tetmajer equation; whereas when E_{sec}/E increases the σ–λ relation is close to the Newlin-Gahagan equation. The values of E_{sec} in Table 1 were substituted into Eq. (15), and the predictions are shown in Fig. 2. The predictions by Eq. (15) are in good agreement with the experimental data for both species. Therefore we believe that the proposed equation can predict the buckling stresses of intermediate wooden columns with various stress–strain relations in the plastic region.

Conclusion

We conducted several buckling tests of spruce and buna columns with various slenderness ratios. Based on the results we formulated an equation for predicting the buckling stress of an intermediate wooden column. The following results were obtained.

1. For both species, the buckling stress of a column with an intermediate slenderness ratio was predicted properly by the tangent modulus theory when the stress–strain relation was derived by the logarithmic function.
2. The buckling stress/slenderness ratio of spruce was convex in the intermediate column range, a result predicted by the Newlin-Gahagan equation. In contrast, the σ–λ relation of buna was linear in the short column range, a tendency close to that predicted by the Tetmajer equation.
3. By introducing the secant modulus into a power function, we propose an equation that can predict the σ–λ relations of intermediate wooden columns with various stress–strain characters in the plastic strain range.

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