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ORIGINAL ARTICLE

Han-Min Park · Masami Fushitani · Hee-Seop Byeon

Derivation and application of an equation for calculating shear modulus of three-ply laminated material beam from shear moduli of individual laminae

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Abstract In a detailed study of the relation between the deflection caused by shear force and the constitution of a laminated material beam, we derived an equation for calculating the shear modulus of a laminated material beam from the shear moduli of individual laminae. The validity of the derived equation was investigated using cross-laminated wood beams made with five species. The calculated shear moduli parallel to the grain of face laminae ranged from 48.3 MPa to 351 MPa, while those perpendicular to the grain of face laminae ranged from 58.0 MPa to 350 MPa. The calculated shear moduli increased markedly with increasing shear modulus in a cross section of perpendicular-direction lamina of a cross-laminated wood beam. The calculated apparent modulus of elasticity (MOE) of cross-laminated wood beams agreed fairly well with the measured apparent MOE values. This fact indicated that the apparent MOE of cross-laminated wood beam was able to be calculated from the true MOE values and shear moduli of individual laminae. The percentage of deflection caused by shear force obtained from the calculated apparent MOE (Y_{sc}) was close to that obtained from the measured apparent MOE (Y_s) and there was a high correlation between both values. From the above results, it was concluded that the derived equation had high validity in calculation of shear modulus of a cross-laminated wood beam.

Key words Three-ply laminated material beam · Shear modulus · Modulus of elasticity · Shear force · Deflection

H.-M. Park (✉) · H.-S. Byeon
College of Agriculture, Institute of Agriculture and Life Science, Gyeongsang National University, Jinju 660-701, Republic of Korea
Tel. +82-55-751-5503; Fax +82-55-762-5999
e-mail: phm0691@gnu.ac.kr

M. Fushitani
Faculty of Agriculture, Tokyo University of Agriculture and Technology, Fuchu, Tokyo 183-8509, Japan

Introduction

When loads are applied vertically to a beam, bending moment and shear force are exerted on it. In the case of four-point loading, shear force and bending moment are exerted between the loading point and the supporting point. The deflection caused by shear force is contained when the deflection at midspan between the supporting points is measured as specified in Japanese Agricultural Standards for glued laminated wood[1] and laminated veneer lumber.[2] It was found by the authors[3,4] that the modulus of elasticity (MOE) parallel to the grain of face lamina of three-ply cross-laminated wood made with sugi wood was markedly influenced by the deflection caused by shear force. Furthermore, to improve the strength performances of cross-laminated wood beams, parallel-laminated and cross-laminated wood beams were manufactured from five species with different densities and shear moduli in cross section, and we investigated the effect of deflection caused by shear force on the static bending strength performance. As a result, it was found that the modulus of rupture as well as the MOE parallel to the grain of face laminae decreased owing to an increase in deflection caused by shear force with decreasing shear modulus in the cross section of the core lamina.[5]

Apparent deflection is composed of the deflections caused by bending moment and shear force, and the percentage of deflection caused by shear force is in proportion to MOE/shear modulus (E/G). Because E parallel to the grain of face laminae is great and G is very small for three-ply cross-laminated wood beams made with sugi,[3–5] E/G is markedly increased. For sugi wood–aluminum hybrid composite beams,[6] E/G is also increased because E is considerably large and G is not.

The deflection caused by shear force of laminated material beams was obtained by subtracting the deflection caused by bending moment from the apparent deflection, and the percentage of deflection caused by shear force was investigated. Furthermore, to adopt a detailed approach to the relation between the deflection caused by shear force and the constitution of the laminated material beam, it is necessary to derive an equation for calculating the shear modulus of the laminated material beam from the shear moduli of individual laminae, like the MOE.

In this study, we derived an equation for calculating the shear modulus of a three-ply laminated material beam from shear moduli of individual laminae. The validity of the derived equation was investigated using cross-laminated wood beams made with five species, which we reported in a previous article.[5]

Derivation of equation to calculate shear modulus of three-ply laminated material beam

Shear stress

Figure 1 shows the configuration of cross section for obtaining the shear stress and shear strain of a three-ply laminated material beam. Shear stress τ that occurs in a three-ply laminated material beam is expressed by the following equation:[7]

$$\tau = \frac{F\sum_{i=1}^{r}(E_iS_i)}{b\sum_{i=1}^{3}(E_iI_i)} \tag{1}$$

where F is the shear force; E_i is the MOE of the ith lamina; S_i is the geometrical moment of area of cross section of the ith lamina about the neutral axis NN; b is the width of lamina; I_i is the geometrical moment of inertia of cross section of the ith lamina; $\tau = \sum_{i=1}^{r}(E_iS_i)$ means the sum only for the layer lower than horizontal line aa' of cross section in which we want to obtain shear stress τ.

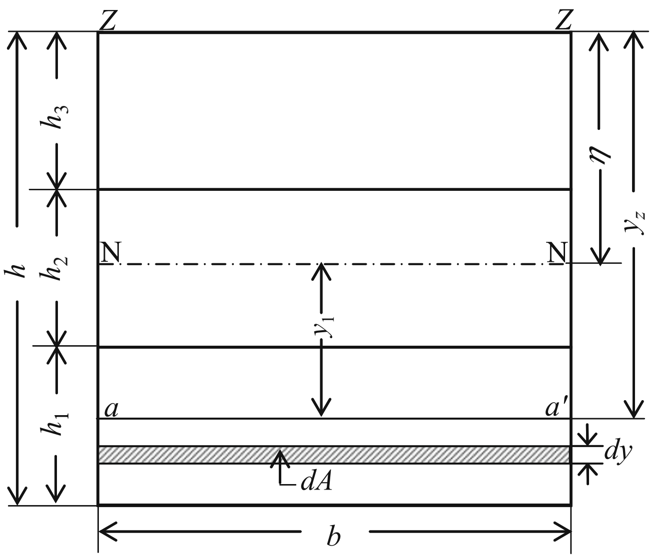


Fig. 1. Configuration of cross section for obtaining shear stress and shear strain of a three-ply laminated material beam. *NN*, neutral axis; η, distance from *ZZ* axis to neutral axis; y_1, distance from neutral axis to aa' horizontal line; y_z, distance from ZZ axis to aa' horizontal line; h, height of three-ply laminated material beam; h_1, h_2, and h_3, heights of individual laminae of three-ply laminated material beam; b, width of three-ply laminated material beam; A, area

The distance from the base axis ZZ to neutral axis NN (η) is given by the following equation:[7]

$$\eta = \frac{\sum_{i=1}^{3}E_i(S_i)_z}{\sum_{i=1}^{3}E_iA_i} \tag{2}$$

where $(S_i)_z$ is the geometrical moment of area of cross section of the ith lamina about the base axis ZZ; A_i is the cross-sectional area of the ith lamina. Hence, from Eq. 2, the distance from the base axis ZZ to neutral axis NN (η) can be derived as

$$\eta = \frac{1}{2}\frac{E_1(2hh_1 - h_1^2) + E_2(2hh_2 - 2h_1h_2 - h_2^2) + E_3(h^2 - 2h_1h - 2h_2h + h_1^2 + 2h_1h_2 + h_2^2)}{E_1h_1 + E_2h_2 + E_3(h - h_1 - h_2)} \tag{3}$$

where E_1, E_2, and E_3 are the MOE values of individual laminae of a laminated material beam; h_1, h_2, and h_3 are the heights of individual laminae; h is the height of the laminated material beam.

When y_1 (distance from neutral axis NN) is in the range of $h - \eta - h_1$ to $h - \eta$ (tension side), shear stress τ is derived from Eq. 1 as follows. The numerator of Eq. 1 is represented as Eq. 4 and the denominator of Eq. 1 is represented as Eq. 5.

$$F\sum_{i=1}^{r}E_iS_i = FE_1S_{1r} = FE_1\int_{y_1}^{h-\eta} bydy \tag{4}$$

$$b\sum_{i=1}^{3}E_iI_i = bE\cdot\frac{bh^3}{12} \tag{5}$$

where E is the MOE of the laminated material beam and E_1 is the MOE of the bottom lamina. Therefore, the shear stress τ in this range can be expressed as Eq. 6 from Eqs. 4 and 5.

$$\tau = \frac{6FE_1}{bh^3E}\left[(h-\eta)^2 - y_1^2\right] \tag{6}$$

When y_1 is in the range of 0 to $h - \eta - h_1$ (tension side), shear stress τ is derived from Eq. 1 as follows. The numerator of Eq. 1 is represented as Eq. 7, and S_1 and S_{2r} of Eq. 7 are represented as Eqs. 8 and 9, respectively.

$$F\sum_{i=1}^{r}E_iS_i = F(E_1S_1 + E_2S_{2r}) \tag{7}$$

where E_2 is the MOE of the core lamina.

$$S_1 = \int_{h-\eta-h_1}^{h-\eta} bydy \tag{8}$$

$$S_{2r} = \int_{y_1}^{h-\eta-h_1} bydy \tag{9}$$

The shear stress τ in this range can be expressed as Eq. 10 from Eqs. 7, 8, 9, and 5.

$$\tau = \frac{6F}{bh^3E}\left\{E_1\left[(h-\eta)^2 - (h-\eta-h_1)^2\right] + E_2\left[(h-\eta-h_1)^2 - y_1^2\right]\right\} \tag{10}$$

When y_1 is in the range of $\eta - h_3$ to η (compression side), shear stress τ is derived from Eq. 1 as follows. The numerator of Eq. 1 is represented as Eq. 11.

$$F\sum_{i=1}^{r} E_i S_i = FE_3 S_{3r} = FE_3\int_{y_1}^{\eta} bydy \tag{11}$$

where E_3 is the MOE of the top lamina. From Eqs. 11 and 5, shear stress τ can be expressed as Eq. 12.

$$\tau = \frac{6FE_3}{bh^3E}\left(\eta^2 - y_1^2\right) \tag{12}$$

When y_1 is in the range of 0 to $\eta - h_3$ (compression side), shear stress τ is derived from Eq. 1 as follows. The numerator of Eq. 1 is represented as Eq. 13 and S_3 and S_{2r} of Eq. 13 are represented as Eqs. 14 and 15, respectively.

$$F\sum_{i=1}^{r} E_i S_i = F\left(E_3 S_3 + E_2 S_{2r}\right) \tag{13}$$

$$S_3 = \int_{\eta-h_3}^{\eta} bydy \tag{14}$$

$$S_{2r} = \int_{y_1}^{\eta-h_3} bydy \tag{15}$$

The shear stress τ in this range can be expressed as Eq. 16 from Eqs. 13, 14, 15, and 5.

$$\tau = \frac{6F}{bh^3E}\left\{E_3\left[\eta^2 - (\eta - h_3)^2\right] + E_2\left[(\eta - h_3)^2 - y_1^2\right]\right\} \tag{16}$$

Shear strain

Shear strain γ is defined in the following equation as a function of shear stress when the shear strain γ is in proportion to shear stress τ.

$$\gamma = \frac{1}{G}\tau \tag{17}$$

where G is shear modulus. Substituting shear stress τ of Eqs. 6, 10, 12, and 16 in Eq. 17, respectively, we can obtain shear strain in each range.

Figure 2 shows schematic diagrams of shear stress and shear strain distribution in longitudinal section of a three-ply laminated material beam. In order to obtain the averaged value of shear strain occurring in the section of a three-ply laminated material beam, we determine the integral value of shear strain distribution in its section.

The integral value of shear strain γdy_1 for dy_1 is obtained as the following equation from Eq. 17.

$$\gamma dy_1 = \frac{1}{G}\tau dy_1 \tag{18}$$

The integral value of shear strain distribution (q_1) from $h - \eta - h_1$ to $h - \eta$ of a three-ply laminated material beam shown in Fig. 2 is represented as follows:

$$q_1 = \int_{h-\eta-h_1}^{h-\eta} \gamma dy_1 = \frac{1}{G_1}\int_{h-\eta-h_1}^{h-\eta} \tau dy_1 \tag{19}$$

where G_1 is the shear modulus of the bottom lamina. Substituting shear stress τ of Eq. 6 in Eq. 19, we obtain

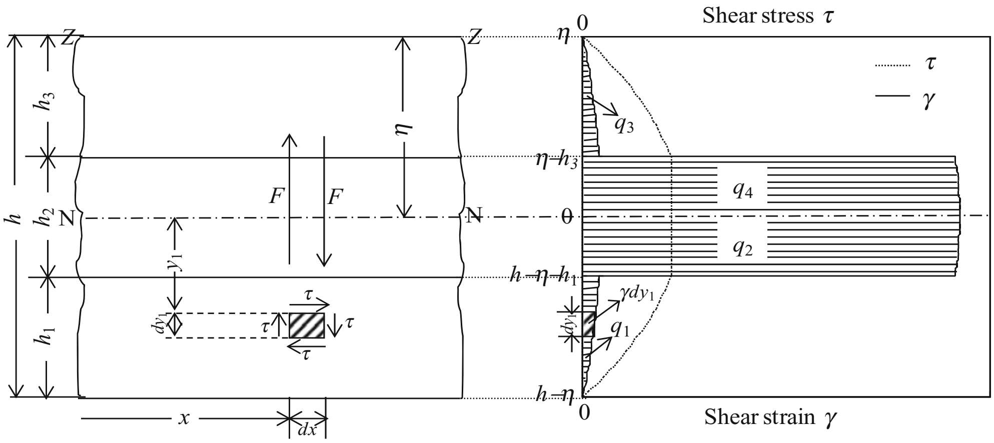
$$q_1 = \frac{6FE_1}{bh^3EG_1}\left[(h-\eta)h_1^2 - \frac{h_1^3}{3}\right] \tag{20}$$

q_2 is the integral value of shear strain distribution from 0 to $h - \eta - h_1$ and is represented as follows:

$$q_2 = \int_0^{h-\eta-h_1} \gamma \mathrm{d}y_1 = \frac{1}{G_2}\int_0^{h-\eta-h_1} \tau dy_1 \tag{21}$$

where G_2 is the shear modulus of the core lamina. Substituting shear stress τ of Eq. 10 in Eq. 21, we obtain

$$q_2 = \frac{6F}{bh^3EG_2}\left\{E_1\left[2(h-\eta)^2 h_1 - 3(h-\eta)h_1^2 + h_1^3\right] + \frac{2}{3}E_2(h-\eta-h_1)^3\right\} \tag{22}$$



(A) (B)

Fig. 2A, B. Schematic diagrams of shear stress and shear strain in longitudinal section of three-ply laminated material beam. *NN*, neutral axis; η, distance from *ZZ* axis to neutral axis; y_1, distance from neutral axis; *F*, shear force; τ, shear stress; γ, shear strain; *h*, height of three-ply laminated material beam; h_1, h_2, and h_3, heights of individual laminae; q_1, q_2, q_3, and q_4, integral values of shear strain distribution from $h - \eta - h_1$ to $h - \eta$, from 0 to $h - \eta - h_1$, from $\eta - h_3$ to η, and from 0 to $\eta - h_3$, respectively. **B** Typical example of shear stress and shear strain distributions parallel to grain of face laminae of cross-laminated wood beam

q_3 is the integral value of shear strain distribution from $\eta - h_3$ to η and is represented as follows:

$$q_3 = \int_{\eta-h_3}^{\eta} \gamma dy_1 = \frac{1}{G_3}\int_{\eta-h_3}^{\eta} \tau dy_1 \tag{23}$$

where G_3 is the shear modulus of the top lamina. Substituting shear stress τ of Eq. 12 in Eq. 23, we obtain

$$q_3 = \frac{6FE_3}{bh^3EG_3}\left(\eta h_3^2 - \frac{h_3^3}{3}\right) \tag{24}$$

q_4 is the integral value of shear strain distribution from 0 to $\eta - h_3$ and is represented as follows:

$$q_4 = \int_0^{\eta-h_3} \gamma dy_1 = \frac{1}{G_2}\int_0^{\eta-h_3} \tau dy_1 \tag{25}$$

Substituting shear stress τ of Eq. 16 in Eq. 25, we obtain

$$q_4 = \frac{6F}{bh^3EG_2}\left[E_3\left(2\eta^2 h_3 - 3\eta h_3^2 + h_3^3\right) + \frac{2E_2(\eta - h_3)^3}{3}\right] \tag{26}$$

q is the total integral value of shear strain distribution from $h - \eta$ to η and can be expressed as the following equation:

$$q = q_1 + q_2 + q_3 + q_4 \tag{27}$$

Therefore,

$$\begin{aligned} q = \frac{6F}{bh^3E}\Bigg\{ & \frac{E_1}{G_1}\left[(h-\eta)h_1^2 - \frac{h_1^3}{3}\right] + \\ & \frac{E_1}{G_2}\left[2(h-\eta)^2 h_1 - 3(h-\eta)h_1^2 + h_1^3\right] + \\ & \frac{2E_2}{3G_2}(h-\eta-h_1)^3 + \frac{E_3}{G_3}\left(\eta h_3^2 - \frac{h_3^3}{3}\right) + \\ & \frac{E_3}{G_2}\left(2\eta^2 h_3 - 3\eta h_3^2 + h_3^3\right) + \frac{2E_2}{3G_2}(\eta - h_3)^3 \Bigg\} \end{aligned} \tag{28}$$

Because q is the total integral value of shear strain distribution for a height of h, the average strain $\bar{\gamma}$ is expressed as Eq. 29.

$$\begin{aligned} \bar{\gamma} = \frac{q}{h} = \frac{1}{h}\left(\frac{6F}{bh^3E}\right)\Bigg\{ & \frac{E_1}{G_1}\left[(h-\eta)h_1^2 - \frac{h_1^3}{3}\right] \\ & + \frac{E_1}{G_2}\left[2(h-\eta)^2 h_1 - 3(h-\eta)h_1^2 + h_1^3\right] + \frac{2E_2}{3G_2}(h-\eta-h_1)^3 \\ & + \frac{E_3}{G_3}\left(\eta h_3^2 - \frac{h_3^3}{3}\right) + \frac{E_3}{G_2}\left(2\eta^2 h_3 - 3\eta h_3^2 + h_3^3\right) + \frac{2E_2}{3G_2}(\eta - h_3)^3 \Bigg\} \end{aligned} \tag{29}$$

Because the averaged shear stress is $\bar{\tau} = F/bh$, shear force can be modified as $F = bh\bar{\tau}$. Therefore, $\bar{\gamma}$ is derived as Eq. 30.

$$\begin{aligned} \bar{\gamma} = \frac{6\bar{\tau}}{h^3E}\Bigg\{ & \frac{E_1}{G_1}\left[(h-\eta)h_1^2 - \frac{h_1^3}{3}\right] \\ & + \frac{E_1}{G_2}\left[2(h-\eta)^2 h_1 - 3(h-\eta)h_1^2 + h_1^3\right] + \frac{2E_2}{3G_2}(h-\eta-h_1)^3 \\ & + \frac{E_3}{G_3}\left(\eta h_3^2 - \frac{h_3^3}{3}\right) + \frac{E_3}{G_2}\left(2\eta^2 h_3 - 3\eta h_3^2 + h_3^3\right) + \frac{2E_2}{3G_2}(\eta - h_3)^3 \Bigg\} \end{aligned} \tag{30}$$

Shear modulus

With the shear modulus of a three-ply laminated material beam being represented as $G = \bar{\tau}/\bar{\gamma}$, shear modulus G from Eq. 30 is derived as Eq. 31.

$$G = \frac{h^3E}{6X} \tag{31}$$

where $X = \frac{E_1}{G_1}\left[(h-\eta)h_1^2 - \frac{h_1^3}{3}\right] + \frac{E_1}{G_2}\left[2(h-\eta)^2 h_1 - 3(h-\eta)h_1^2 + h_1^3\right] + \frac{2}{3}\cdot\frac{E_2}{G_2}(h-\eta-h_1)^3 + \frac{2}{3}\cdot\frac{E_2}{G_2}(\eta-h_3)^3 + \frac{E_3}{G_2}(2\eta^2 h_3 - 3\eta h_3^2 + h_3^3) + \frac{E_3}{G_3}(\eta h_3^2 - \frac{h_3^3}{3})$.

Application of derived equation to calculation of shear modulus of three-ply cross-laminated wood beam

Specimen preparation

Species used for three-ply laminated wood beams included two softwoods: sugi (Japanese cedar, *Cryptomeria japonica* D. Don) and hinoki (Japanese cypress, *Chamaecyparis obtusa* Endl.); and three hardwoods: kiri (royal paulownia, *Paulownia tomentosa* Steud.), katsura (katsura, *Cercidiphyllum japonicum* Sieb. et Zucc.), and buna (beech, *Fagus crenata* Blume) as described in the previous article.[5] A resorcinol–phenol resin-type adhesive formulated for a room-temperature cure was used, and the amount of spread was 300 g/m^2. Figure 3 shows the three-ply parallel-laminated and cross-laminated wood beam specimens tested.

$P_{\parallel}$ types are the laminated wood beam specimens with all layers composed of laminae parallel to the grain [$P_{\parallel}$(KI), $P_{\parallel}$(SU), $P_{\parallel}$(HI), $P_{\parallel}$(KA), $P_{\parallel}$(BU); KI, kiri; SU, sugi; HI, hinoki; KA, katsura; BU, buna]. $P_{\perp}$ types are the laminated wood beam specimens with all layers composed of laminae perpendicular to the grain [$P_{\perp}$(KI), $P_{\perp}$(SU), $P_{\perp}$(HI), $P_{\perp}$(KA), $P_{\perp}$(BU)].

$C_{\parallel}$(S) type was composed of longitudinal-direction laminae of sugi in the faces and perpendicular-direction lamina of five species in the core [$C_{\parallel}$(SKI), $C_{\parallel}$(SSU), $C_{\parallel}$(SHI), $C_{\parallel}$(SKA), and $C_{\parallel}$(SBU)], where prefix "S" means that the longitudinal-direction laminae in the faces were made of sugi wood. $C_{\parallel}$(B) type was composed of longitudinal-direction laminae of buna in the faces and perpendicular-direction lamina of five species in the core [$C_{\parallel}$(BKI), $C_{\parallel}$(BSU), $C_{\parallel}$(BHI), $C_{\parallel}$(BKA), and $C_{\parallel}$(BBU)], where prefix "B" means that longitudinal-direction laminae in the faces were made of buna wood. $C_{\perp}$(S) type was composed of perpendicular-direction laminae of five species in the faces and longitudinal-direction lamina of sugi in the core [$C_{\perp}$(SKI), $C_{\perp}$(SSU), $C_{\perp}$(SHI), $C_{\perp}$(SKA), and $C_{\perp}$ (SBU)], where the prefix "S" means that the longitudinal-direction lamina in the core was made of sugi wood. $C_{\perp}$(B) type was composed of perpendicular-direction laminae of five species in the faces and longitudinal-direction lamina of buna in the core [$C_{\perp}$(BKI), $C_{\perp}$(BSU), $C_{\perp}$(BHI), $C_{\perp}$(BKA),

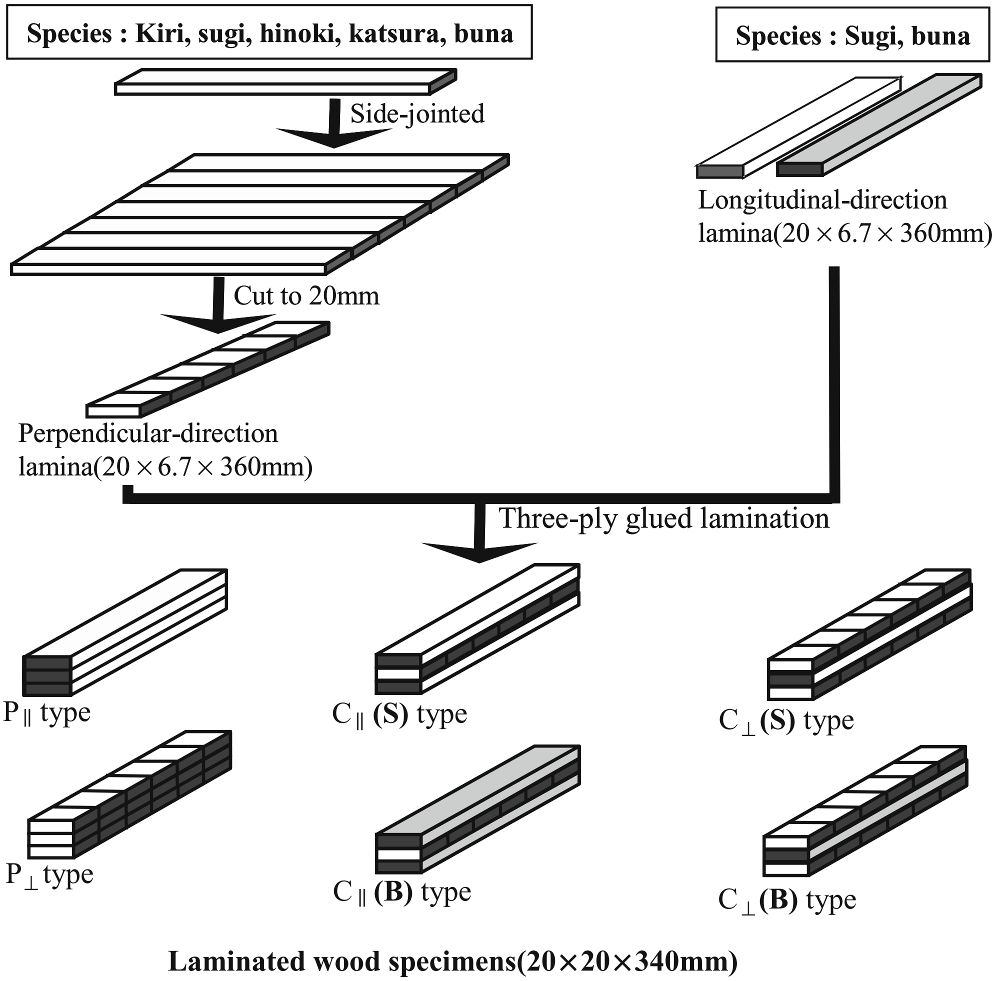


Fig. 3. Parallel-laminated and cross-laminated wood specimens. *S*, sugi; *B*, buna used as longitudinal-direction laminae in cross-laminated woods

and $C_{\perp}$(BBU)], where the prefix "B" means that the longitudinal-direction lamina in the core was made of buna wood.

$P_{\parallel}$ types and $P_{\perp}$ types were the specimens used to measure the bending strength properties parallel and perpendicular to the grain of parallel-laminated woods, respectively. $C_{\parallel}$ types and $C_{\perp}$ types were the specimens used to measure the bending strengths parallel and perpendicular to the grain of the face laminae of cross-laminated woods, respectively. The number of specimens per type was 3 and total number of specimens was 90.

Bending strength properties test

The static bending test for all laminated wood specimens was conducted by four-point loading. The span was 300 mm, the distance between loading points and supporting points was 100 mm, and the cross head speed was set at 5.0 mm/min. The deflection at midspan was measured with a dial gauge, and load–deflection curves were recorded with an X-Y recorder.

Shear modulus measurement

Static three-point bending test was conducted to obtain shear modulus (G) for parallel-laminated wood beam specimens. The relation between the square of the height/span ratio $[(h/l)^2]$ and the compliance ($1/E_{\alpha}$) is expressed by the following equation:[8]

$$\frac{1}{E_{\alpha}} = \frac{1}{E} + \frac{k}{G}\left(\frac{h}{l}\right)^2 \tag{32}$$

where E_{α} is the measured MOE, E is the true MOE, G is the shear modulus, k is 6/5 in the case of a rectangular cross section,[9] and h and l are the height and span of the specimen.

The span was decreased from 300 mm to 120 mm in decrements of 20 mm, and the MOE corresponding to each span was calculated. The regression line was described from the relation between the square of the height/span ratio and the compliance ($1/E_{\alpha}$), and the regression equations were obtained. By applying the regression equation to Eq. 32, shear moduli (G) and true MOE (E) for $P_{\parallel}$ and $P_{\perp}$ types were calculated and the E/G values were obtained.

Results and discussion

Calculated value of shear modulus

The calculated values of shear moduli of cross-laminated wood beams were obtained using the derived equation for calculating shear modulus of a three-ply laminated material beam. Table 1 shows the shear moduli (G_1, G_2, and G_3) and true MOEs (E_1, E_2, and E_3) of individual laminae and the true MOEs of cross-laminated wood beams that were used for calculating the shear modulus of a cross-laminated wood beam.

The calculated values of shear moduli of cross-laminated wood beams are shown in Table 2. For the $C_\perp$(S) type, the calculated value of shear modulus for buna [$C_\perp$(SBU)] had the highest value (322 MPa), whereas that for sugi [$C_\perp$(SSU)] had the lowest value (58.0 MPa). The values were in the order $C_\perp$(SBU) > $C_\perp$(SKA) > $C_\perp$(SHI) > $C_\perp$(SKI) > $C_\perp$(SSU), and were in the same order as shear moduli of $P_\perp$ type made with five species. By replacing the perpendicular-direction lamina of five species in the core of $P_\perp$ type with the longitudinal-direction lamina of sugi, these values increased to 1.5–2.8 times that for the $P_\perp$ type and the extent of the increase increased with decreasing density of species. It was found that the calculated values of shear moduli for the $C_\perp$(B) type were slightly higher than those for the $C_\perp$(S) type.

In contrast, for the $C_\parallel$(S) type, the calculated values of shear modulus for buna [$C_\parallel$(SBU)] had the highest value (317 MPa), whereas that for sugi [$C_\parallel$(SSU)] had the lowest value (48.3 MPa). The values were in the order $C_\parallel$(SBU) > $C_\parallel$(SKA) > $C_\parallel$(SHI) > $C_\parallel$(SKI) > $C_\parallel$(SSU). By replacing the longitudinal-direction lamina of sugi in the core of the $P_\parallel$(SU) type with the perpendicular-direction lamina of five species, these values decreased to 0.09–0.59 times that for the $P_\parallel$(SU) type and it was found that the extent of the decrease increased with decreasing shear modulus in the cross section of the perpendicular-direction lamina in the core. Also, these values decreased to 0.07–0.48 times that for the $P_\parallel$(BU) type by replacing the longitudinal-direction lamina of buna in the core of the $P_\parallel$(BU) type with the perpendicular-direction lamina of five species.

Apparent deflection y_α of a beam for four-point bending is expressed in the following equation:[3]

$$y_\alpha = y_m + y_s = \frac{Pl_1\left(3l^2 - 4l_1^2\right)}{4bh^3E} + \frac{kPl_1}{2AG} \tag{33}$$

$$= \frac{Pl_1\left(3l^2 - 4l_1^2\right))}{4bh^3E}\left[1 + \frac{2.4h^2}{3l^2 - 4l_1{}^2}\cdot\frac{E}{G}\right]$$

where y_m is the deflection caused by bending moment; y_s is the deflection caused by shear force; E is the true MOE; G is the shear modulus; P is the applied load; b and h are the width and height of the beam (20 mm, respectively); l is the span (300 mm); l_1 is the distance between a loading point and a supporting point (l/3); and k is 6/5 in the case of a rectangular cross section.[9]

From Eq. 33, the percentage of deflection caused by shear force is in proportion to E/G. Thus, E_γ/G_c values were obtained from the calculated values of shear modulus (G_c) mentioned above and true MOE (E_γ) reported in the previous report,[5] and are shown in Table 2.

Table 1. Shear moduli and moduli of elasticity of individual laminae, and modulus of elasticity of cross-laminated wood beam, used for calculation of shear modulus of cross-laminated wood beam

Type[a]	G_1 (MPa)	G_2 (MPa)	G_3 (MPa)	E_1 (GPa)	E_2 (GPa)	E_3 (GPa)	E_γ (GPa)
$C_\parallel$(SKI)	533	40.2	533	9.66	0.537	9.66	9.29
$C_\parallel$(SSU)	533	23.6	533	9.66	0.833	9.66	9.29
$C_\parallel$(SHI)	533	56.4	533	9.66	0.928	9.66	9.35
$C_\parallel$(SKA)	533	170	533	9.66	1.36	9.66	9.40
$C_\parallel$(SBU)	533	215	533	9.66	1.65	9.66	9.39
$C_\parallel$(BKI)	726	40.2	726	9.18	0.537	9.18	8.85
$C_\parallel$(BSU)	726	23.6	726	9.18	0.833	9.18	8.92
$C_\parallel$(BHI)	726	56.4	726	9.18	0.928	9.18	9.01
$C_\parallel$(BKA)	726	170	726	9.18	1.36	9.18	9.07
$C_\parallel$(BBU)	726	215	726	9.18	1.65	9.18	9.05
$C_\perp$(SKI)	40.2	533	40.2	0.537	9.66	0.537	0.906
$C_\perp$(SSU)	23.6	533	23.6	0.833	9.66	0.833	1.14
$C_\perp$(SHI)	56.4	533	56.4	0.928	9.66	0.928	1.20
$C_\perp$(SKA)	170	533	170	1.36	9.66	1.36	1.66
$C_\perp$(SBU)	215	533	215	1.65	9.66	1.65	1.94
$C_\perp$(BKI)	40.2	726	40.2	0.537	9.18	0.537	0.898
$C_\perp$(BSU)	23.6	726	23.6	0.833	9.18	0.833	1.13
$C_\perp$(BHI)	56.4	726	56.4	0.928	9.18	0.928	1.20
$C_\perp$(BKA)	170	726	170	1.36	9.18	1.36	1.65
$C_\perp$(BBU)	215	726	215	1.65	9.18	1.65	1.91

Each value is the average of three measurements and is also shown in Table 2. $h_1 = h_2 = h_3 = 6.7$ mm

G_1, G_2, and G_3, shear moduli of bottom, core, and top laminae; E_1, E_2, and E_3, true modulus of elasticity (MOE) values of bottom, core, and top laminae; E_γ, true MOE of cross-laminated wood

[a] Prefix S, sugi; prefix B, buna used as longitudinal-direction laminae of cross-laminated woods

KI, kiri; SU, sugi; HI, hinoki; KA, katsura; BU, buna

Table 2. Shear modulus calculated by using derived equation (G_c), apparent modulus of elasticity calculated from G_c ($E_{\alpha c}$), E_γ/G_c, and percentage of calculated deflection caused by shear force (Y_{sc}) of cross-laminated wood beams

Type	G_c (MPa)	G (MPa)	$E_{\alpha c}$ (GPa)	E_α (GPa)	E_β (GPa)	E_γ (GPa)	E_γ/G_c	E_γ/G	Y_{sc} (%)	Y_s (%)
$P_\parallel$(KI)	–	284	–	5.69	5.73	6.21	–	21.9	–	8.5
$P_\parallel$(SU)	–	533	–	8.98	9.19	9.66	–	18.1	–	7.1
$P_\parallel$(HI)	–	750	–	12.5	12.9	13.5	–	18.0	–	7.1
$P_\parallel$(KA)	–	864	–	8.80	8.81	9.21	–	10.7	–	4.5
$P_\parallel$(BU)	–	726	–	8.67	8.96	9.18	–	12.6	–	5.6
$P_\perp$(KI)	–	40.2	–	0.510	0.470	0.537	–	13.4	–	5.0
$P_\perp$(SU)	–	23.6	–	0.735	0.697	0.833	–	35.3	–	11.8
$P_\perp$(HI)	–	56.4	–	0.872	0.774	0.928	–	16.5	–	6.0
$P_\perp$(KA)	–	170	–	1.31	1.23	1.36	–	8.00	–	3.1
$P_\perp$(BU)	–	215	–	1.60	1.48	1.65	–	7.67	–	3.0
$C_\parallel$(SKI)	79.6	–	6.24	5.83	8.75	9.29	117	–	32.8	37.3
$C_\parallel$(SSU)	48.3	–	5.16	5.68	8.75	9.29	192	–	44.5	38.9
$C_\parallel$(SHI)	109	–	6.89	6.25	8.80	9.35	85.8	–	26.3	33.1
$C_\parallel$(SKA)	269	–	8.21	7.52	8.85	9.40	34.9	–	12.7	20.0
$C_\parallel$(SBU)	317	–	8.36	7.68	8.84	9.39	29.6	–	11.0	18.2
$C_\parallel$(BKI)	81.5	–	6.08	5.78	8.33	8.85	109	–	31.3	34.7
$C_\parallel$(BSU)	49.4	–	5.08	5.31	8.39	8.92	181	–	43.0	40.5
$C_\parallel$(BHI)	113	–	6.76	6.06	8.48	9.01	79.7	–	25.0	32.8
$C_\parallel$(BKA)	294	–	8.03	7.08	8.54	9.07	30.9	–	11.5	21.9
$C_\parallel$(BBU)	351	–	8.17	7.60	8.52	9.05	25.8	–	9.7	16.1
$C_\perp$(SKI)	113	–	0.877	0.787	0.794	0.906	8.02	–	3.2	13.1
$C_\perp$(SSU)	58.0	–	1.05	1.06	1.01	1.14	19.7	–	7.9	7.1
$C_\perp$(SHI)	120	–	1.15	1.14	1.07	1.20	10.0	–	4.2	4.9
$C_\perp$(SKA)	279	–	1.62	1.59	1.51	1.66	5.95	–	2.4	4.6
$C_\perp$(SBU)	322	–	1.89	1.82	1.77	1.94	6.02	–	2.6	6.2
$C_\perp$(BKI)	116	–	0.870	0.821	0.786	0.898	7.74	–	3.1	8.6
$C_\perp$(BSU)	58.7	–	1.05	1.04	1.01	1.13	19.3	–	7.1	8.6
$C_\perp$(BHI)	125	–	1.15	1.15	1.07	1.20	9.60	–	4.2	4.0
$C_\perp$(BKA)	303	–	1.61	1.60	1.50	1.65	5.45	–	2.4	3.3
$C_\perp$(BBU)	350	–	1.87	1.84	1.75	1.91	5.46	–	2.1	3.9

Each value of G, E_α, E_β, and E_γ is the average value of three measurements. G, E_α, E_β, and E_γ were reported previously by Park et al.[5]
G, Shear modulus of parallel-laminated wood beam; E_α, measured apparent MOE; E_β, MOE calculated from true MOE values of individual laminae using equivalent cross-section method; E_γ, true MOE of laminated wood beam; Y_s, percentage of measured deflection caused by shear force versus total deflection

The true MOE (E_γ) of cross-laminated wood beam was obtained as follows. The true MOE (E_γ) of parallel-laminated wood beam was calculated using $E = E_\alpha\,(1 + \varphi)$ [$\varphi = 2.4h^2/(3l^2 - 4l_1^2)\cdot(E/G) = 0.00417\cdot(E/G)$] from E_α and E/G. E/G was obtained by the method described for shear modulus measurement. The apparent MOEs of individual laminae were measured, and the true MOEs of individual laminae were calculated from E_α and E/G. The true MOE (E_β) of parallel-laminated wood beams without the contribution of the glue line was calculated from the true MOEs of the individual laminae by the equivalent cross-section method. Furthermore, the relation between the ratio of the true MOE (E_γ) having the contribution of glue line to E_β ($R_g = E_\gamma/E_\beta$) and $1/E_\beta$ of parallel-laminated wood beams was represented by the following equation:

$$R_g = 0.068\cdot(1/E_\beta) + 1.054\,(r = 0.742,\ \text{significant at 1\% level})$$

E_β of cross-laminated wood beams was calculated from the true MOEs of individual laminae using the equivalent cross-section method. Assuming that this regression equation of parallel-laminated wood beam was also valid for cross-laminated wood beam, then R_g of cross-laminated wood beam was obtained from E_β by the regression equation and E_γ was obtained from R_g and E_β.

For $C_\perp$(S) type, the E_γ/G_c values were in the order $C_\perp$(SSU) > $C_\perp$(SHI) > $C_\perp$(SKI) > $C_\perp$(SBU) > $C_\perp$(SKA), and the values had low values of 5.95–19.7, and its difference among species of laminae used in the faces was found to be small. By replacing the perpendicular-direction lamina of five species in the core of $P_\perp$ type with the longitudinal-direction lamina of sugi, E_γ and G_c increased to 1.2–1.7 times and 1.5–2.8 times those for the $P_\perp$ type, respectively, and E_γ/G_c decreased to 0.56–0.78 times that for the $P_\perp$ type. A similar result was found for $C_\perp$(B) type.

In contrast, for the $C_\parallel$(S) type, the E_γ/G_c values were in the order $C_\parallel$(SSU) > $C_\parallel$(SKI) > $C_\parallel$(SHI) > $C_\parallel$(SKA) > $C_\parallel$(SBU). Sugi [$C_\parallel$(SSU)] had the highest value (192), whereas buna [$C_\parallel$(SBU)] had the lowest value (29.6). There was a marked difference in the value among species of laminae used in the core. By replacing the longitudinal-direction lamina of sugi in the core of $P_\parallel$(SU) type with the perpendicular-direction lamina of five species, E_γ/G_c increased by 1.6 (buna) to 11 times (sugi). The reason is that E_γ values decreased only to 0.96–0.97, whereas G_c decreased to 0.09 (sugi) to 0.59 (buna). It was found that there was a large difference in E_γ/G_c among species used for the core lamina. E_γ/G_c values were in the order $C_\parallel$(SSU) > $C_\parallel$(SKI) > $C_\parallel$(SHI) > $C_\parallel$(SKA) > $C_\parallel$(SBU). A similar result was found for $C_\parallel$(B) type.

Relation between calculated value and measured value of apparent MOE

The calculated apparent MOE ($E_{\alpha c}$) of cross-laminated wood beam is derived as the following equation from Eq. 33.

$$E_{\alpha c} = E_{\gamma}/(1+\varphi) \tag{34}$$

where E_{γ} is the true MOE, $\varphi = 2.4h^2/(3l^2 - 4l_1^2)\cdot(E_{\gamma}/G_c) = 0.00417\cdot(E_{\gamma}/G_c)$.

Figure 4 shows the percentage of difference between the measured value and the calculated value versus the measured value of apparent MOE of cross-laminated wood beam related to its G_c. For the $C_{\perp}$ type, its difference was in the range of 0.9% to −11%, and the calculated value was in fair agreement with the measured value.

On the other hand, for the $C_{\parallel}$ type, the calculated values of $C_{\parallel}$(SSU) and $C_{\parallel}$(BSU), with core composed of perpendicular-direction lamina of sugi with the lowest shear modulus of five species, were 9.2% and 4.3% smaller than the measured values, respectively. Conversely, for the other $C_{\parallel}$ types, the calculated values were 5.2% to 14% greater than the measured values. The range of the difference between both values was slightly wider in $C_{\parallel}$ type than in $C_{\perp}$ type, but good agreement between both values was also found. From the above results, it was found that the calculated value of cross-laminated wood beams was able to be obtained from the MOEs and shear moduli of laminae.

Percentage of deflection caused by shear force

The percentage of deflection caused by shear force versus total deflection can be obtained by the following equation from the true MOE and the measured apparent MOE as mentioned in the previous report.[5]

$$Y_s = \frac{y_{\alpha} - y_m}{y_{\alpha}} \times 100 = \frac{E_{\gamma} - E_{\alpha}}{E_{\gamma}} \times 100\ (\%) \tag{35}$$

Also, the percentage of deflection caused by shear force versus total deflection of cross-laminated wood beams can be obtained by the following equation from the true MOE and the calculated apparent MOE.

$$Y_{sc} = \frac{y_{\alpha c} - y_m}{y_{\alpha c}} \times 100 = \frac{E_{\gamma} - E_{\alpha c}}{E_{\gamma}} \times 100\ (\%) \tag{36}$$

Table 2 shows the Y_s values that were reported in our previous article[5] and Y_{sc} values, and the comparison of these values is shown in Fig. 5.

For the $C_{\perp}$(S) type, the Y_{sc} value, which is the calculated value of the percentage of deflection caused by shear force, was in the range of 2.4%–7.9%, and the values were slightly lower than Y_s values of 4.6%–13.1% calculated from the measured values in the previous study.[5] However, their differences were small. A similar result was found for $C_{\perp}$(B) type with the core composed of longitudinal-direction lamina of buna.

On the other hand, for the $C_{\parallel}$(S) type, the Y_{sc} of $C_{\parallel}$(SSU) with the core composed of perpendicular-direction lamina of sugi had the highest value of 44.5%, and that of $C_{\parallel}$(SBU) with the core composed of perpendicular-direction lamina of buna had the lowest value of 11.0%. The Y_{sc} value was in the order $C_{\parallel}$(SSU) > $C_{\parallel}$(SKI) > $C_{\parallel}$(SHI) > $C_{\parallel}$(SKA) > $C_{\parallel}$(SBU), and it had the same order as the Y_s described in the previous article.[5] A similar result was found for $C_{\parallel}$(B) type.

Figure 5 shows the regression line for the relation between Y_{sc} and Y_s by the least-squares method and a very high correlation (r = 0.955, significant at 1% level) was obtained. In this figure, the straight line of $y = x$ is shown by the dotted line to examine if Y_{sc} agreed with Y_s. $C_{\perp}$(SKI) and $C_{\perp}$(BKI) with faces composed of perpendicular-

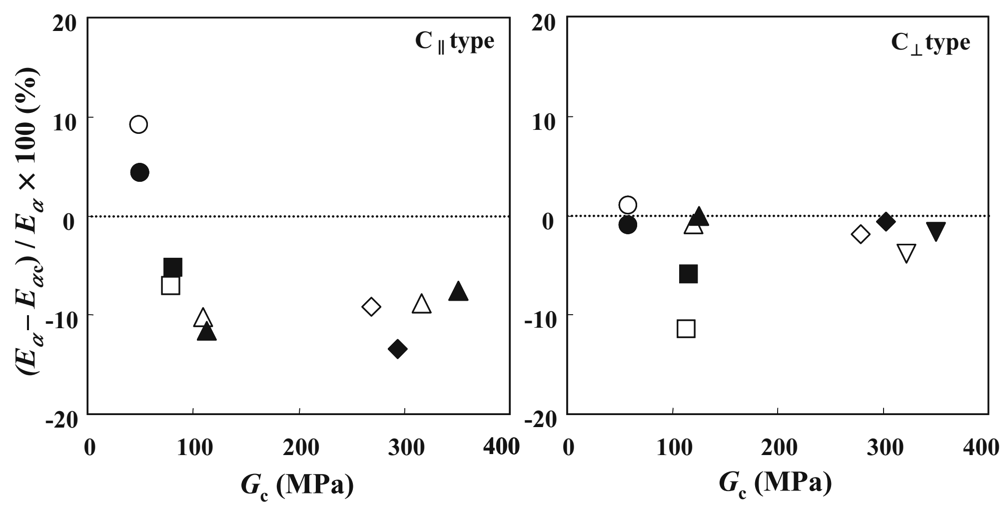


Fig. 4. Percentage of difference between measured value and calculated value versus measured value of apparent modulus of elasticity (MOE) of cross-laminated wood beam related to its G_c. E_{α}, measured apparent MOE; $E_{\alpha c}$, calculated apparent MOE. *Open circles*, SSU; *filled circles*, BSU; *open squares*, SKI; *filled squares*, BKI; *open triangles*, SHI; *filled triangles*, BHI; *open diamonds*, SKA; *filled diamonds*, BKA; *open inverted triangles*, SBU; *filled inverted triangles*, BBU. Prefix *S*, sugi; prefix *B*, buna used as longitudinal-direction laminae of cross-laminated woods. *KI*, kiri; *SU*, sugi; *HI*, hinoki; *KA*, katsura; *BU*, buna

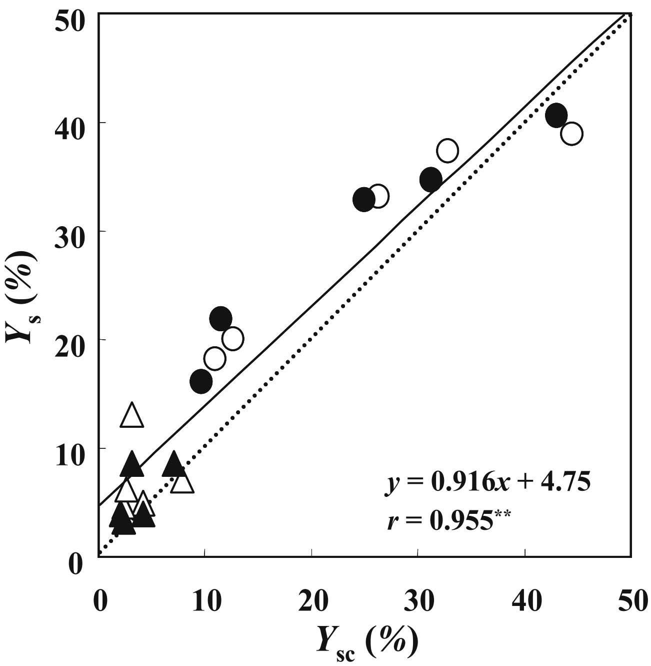


Fig. 5. Comparison of Y_s value with Y_{sc} value of cross-laminated wood beam. Y_s, percentage of deflection caused by shear force and calculated from measured apparent MOE and true MOE; Y_{sc}, percentage of deflection caused by shear force and calculated from calculated apparent MOE and true MOE. Dotted line, $y = x$; *open circles*, $C_{\parallel}$(S) type; *filled circles*, $C_{\parallel}$(B) type; *open triangles*, $C_{\perp}$(S) type; *filled triangles*, $C_{\perp}$(B) type; *double asterisk*, significant at 1% level

direction lamina of kiri were situated somewhat above the line, whereas $C_{\parallel}$(SSU) and $C_{\parallel}$(BSU) with cores composed of perpendicular-direction lamina of sugi were situated a little below it. On the whole, the Y_s values were higher than the Y_{sc} values, but their values were near the line. From the above results, it is considered that the validity of the derived equation for calculating shear modulus was verified.

Conclusions

In this study, we derived an equation for calculating the shear modulus of a three-ply laminated material beam from shear moduli of individual laminae. The validity of the derived equation was investigated using cross-laminated wood beams made with five species reported in our previous article.[5] The key findings of the study are:

1. The calculated shear moduli of cross-laminated wood beams increased markedly with increasing shear modulus in cross section for perpendicular-direction laminae of both core and faces.
2. The calculated value and the measured value of apparent MOE of cross-laminated wood beam were in good agreement, and it was found that the apparent MOE was able to be calculated from the true MOEs and shear moduli of laminae.
3. For a cross-laminated wood beam with perpendicular-direction laminae in the faces, the calculated value and the measured value of the percentage of deflection caused by shear force were small. For beams with longitudinal-direction laminae in the faces, these same values increased markedly with decreasing shear modulus in cross section of core lamina. The Y_s values were close to the Y_{sc} values and there was a very high correlation between them.

From the above results, the high validity of the derived equation for calculating shear modulus of cross-laminated wood beams was apparent.

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