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ORIGINAL ARTICLE

A static analysis of two-shaft columns spaced by gussets

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Abstract The paper focuses on timber constructions. It analyses two-shaft columns spaced by gussets made of timber, plywood, particleboard or fibreboard. Based on the theory authored by Timoshenko and Gere, some formulae, defining the column critical force, its slenderness ratio, the shear force applied to the column and the maximum shear force that a column can carry, were derived. Next, based on the derived formulae and those applied in the literature (Standard EN-1995 Eurocode 5), a comparative analysis was conducted on the load-bearing capacity and gusset calculation for the columns. The calculations demonstrate that there are substantial discrepancies between the static values being compared and both calculation methods lead to partially divergent results.

Keywords Two-shaft columns · Gussets · Shear strain · Critical load-bearing capacity · Shearing in columns

Symbols

Latin letters

a	Initial maximum curvature of a column
a_1	Axial distance of shafts
A	Cross section, cross section of column
A_p	Cross section of gusset
e	Eccentric of force P application
E	Modulus of elasticity, modulus of elasticity of timber
E_{mean}	Mean value of modulus of elasticity
$E_{0.05}$	Fifth-percentile value of modulus of elasticity
E_s	Mean value of modulus of elasticity of shaft
$E_{0.05}$, $E^s_{0.05}$	Fifth-percentile value of modulus of elasticity of shaft
E_p	Mean value of modulus of elasticity of gussets
$E^p_{0.05}$	Fifth-percentile value of modulus of elasticity of gussets
$f_{c,0,k}$	Characteristic compressive strength along the grain
$f_{c,0,d}$	Design compressive strength along the grain
$f_{v,k}$	Characteristic panel shear strength
$f_{r,k}$	Characteristic planar (rolling) shear strength
$f_{m,k}$	Characteristic bending strength
G	Shear modulus
G_{mean}	Mean value of shear modulus, mean modulus of rigidity
G_p	Mean value of shear modulus for gussets
$G^p_{0.05}$	Fifth-percentile value of shear modulus for gussets
i	Radius of gyration of a column treated as solid
i_1	Radius of gyration of a column shaft
I	Second moment of area of a section
I_p	Second moment of area of gussets
I_s	Second moment of area of a column shaft
k_c	Instability factor
l_c	Length of a column treated as solid
l_1	Axial span of gussets
$M(x)$	Bending moment
$M_{\max}$	Maximum bending moment
n	Number of column shafts $n = 2$
n	Load-bearing capacity of a column
P	Compressive force
$q(x)$	Transverse load

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V_p^a Maximum shear force, caused by compressive force P, for a column with its initial curvature described with a sinusoid

V_p^e Maximum shear force, caused by compressive force P, for a column, where force P acts on eccentric e

$V_{p,\max}^a$ Maximum shear force a column can carry, for a column with its initial curvature described with a sinusoid

$V_{p,\max}^e$ Maximum shear force a column can carry, for a column, where force P acts on eccentric e

$y(x)$ Deflection line function

$z_{\max}$ Distance between the neutral axis and the extreme grain

Greek letters

η_{ef} Connection factor for which values are given in Table C.1 (EN-1995 Eurocode 5 [4])

λ Slenderness ratio of a column treated as solid

λ_1 Slenderness ratio of a single column shaft, where some buckling occurs between the gussets

μ Energetic shear coefficient, for a column treated as solid

Introduction

Beside solid columns, two-shaft columns spaced by gussets are a commonly applied structural element. Generally, in timber constructions, solid wood or glued laminated timber are used for the shafts, while the gussets are made of wood, plywood, particleboard or fibreboard. Timber structures, basing on EN 338 [1], feature a relatively high ratio of modulus of elasticity E to shear modulus G, which, for timber class e.g. C18, amounts to $\frac{E}{G} = 16$. Thus, in order to identify critical load, it becomes necessary to take shear strain into consideration. In steel structures, for instance, basing on EN-1993 Eurocode 3 [2], the ratio of modulus of elasticity E to shear modulus G is much lower and amounts to $\frac{E}{G} = 2.5$, which means that, while conducting static analysis for elements produced from that material, the influence of shear strain on load-bearing capacity is relatively marginal and, in many cases, neglectable. The current state of the art, given in Timoshenko and Gere [3], indicates that critical load-bearing capacity of columns with gussets in question is always lower than the critical load-bearing capacity of solid columns with the same cross section and the same slenderness ratio. This results from the fact that the effect of shear on displacement within a column with gussets is much stronger than in a solid column. The author does research on structure stability and he studied the theory given in Timoshenko and Gere [3] dealing with two-shaft columns with gussets taking into consideration the influence of shearing on critical load-bearing capacity. The author found then that the formulae used to date and given in the literature (EN-1995 Eurocode 5 [4]) are too simple to describe, in an accurate way, the static work of the column as a composite structure made not only of columns with wooden gussets but also of different materials of different strength and elasticity. Therefore, a comparative static analysis of the columns with gussets was performed based on formulae provided for by the subject literature (EN-1995 Eurocode 5 [4]), not including a factor that reflects the impact of shear on critical load-bearing capacity of the columns, and some formulae based on the theory presented in Timoshenko and Gere [3] that take shear effect into consideration.

Calculations for two-shaft columns with gussets, according to the subject literature (EN-1995 Eurocode 5 [4]): state of the art

Slenderness ratio of a two-shaft column with gussets fixed rigidly to the shaft, based on the applicable literature (EN-1995 Eurocode 5 [4]), is hereby derived from the formula:

$$\lambda_{ef} = \sqrt{\lambda^2 + \eta_{ef} \cdot \frac{n}{2}\lambda_1^2} \tag{1}$$

According to the literature (EN-1995 Eurocode 5 [4]), shear forces in columns V_p and maximum shear forces $V_{p,\max}$ the column is able to carry, can be derived from these formulae:

$$V_p = \begin{cases} \dfrac{P}{120 \cdot k_c} & \text{for } \lambda_{ef} < 30 \\ \dfrac{P \cdot \lambda_{ef}}{3600 \cdot k_c} & \text{for } 30 \le \lambda_{ef} < 60 \\ \dfrac{P}{60 \cdot k_c} & \text{for } 60 \le \lambda_{ef} \end{cases} \tag{2}$$

In order to determine maximum shear force $V_{p,\max}$ the column is able to carry, the value of maximum force P, at which load-bearing capacity of the column is not exceeded, needs to be brought into formulae (2).

Derivation of formulae for load-bearing column capacity, based on the theory presented in Timoshenko and Gere [3], accounting for the influence of shearing

Determination of critical force

In order to determine critical force, taking shear into account, a rod deflection differential equation, as derived by the author, was used. That equation accounts for the influence of shear forces on deflections, and it has the following form:

$$\frac{d^2y(x)}{dx^2} + k^2y(x) = -\frac{1}{EI\left(1-\frac{\mu P}{GA}\right)}M(x) - \frac{\mu}{GA\left(1-\frac{\mu P}{GA}\right)}q(x) \quad (3)$$

where k^2 equals:

$$k^2 = \frac{P}{EI\left(1-\frac{\mu P}{GA}\right)} \quad (4)$$

Derivation of Eq. (3) is presented in the author's article Śliwka [5].

In that paper, a case of a rod loaded with longitudinal compressive force P was analyzed, which results in the fact that the bending moment $M(x)$, present in the equation, and transverse load $q(x)$ take up the 0 value $M(x)=0$ $q(x)=0$. That yielded the following equation:

$$\frac{d^2(x)}{dx^2} + k^2y(x) = 0 \quad (5)$$

The equation was solved, using the method of operational calculus, based on Laplace transformation Osiowski [6]. The solution of the equation is the following function:

$$y(x) = w_0 \cos kx + w_1 \frac{1}{k} \sin kx \quad (6)$$

Critical force was derived from a boundary condition, saying that the value of the argument of function sin at point $x=l$ $k{\cdot}l$ equals:

$$k \cdot l = n\pi \quad (7)$$

Replacing k in the equation, where $k^2 = \frac{P}{EI\left(1-\frac{\mu P}{GA}\right)}$ and $n=1$, the following formula for critical force $P = P_{c,\mathrm{crit}}$, was obtained:

$$P_{c,\mathrm{crit}} = \frac{P_e}{1 + P_e\frac{\mu}{GA}} \quad (8)$$

where:

$$P_e = \frac{\pi^2 EI}{l_c^2} \quad (9)$$

Formulae (8) and (9) are presented in the author's article Śliwka [5].

Determination of slenderness λ_{ef} for a two-shaft column spaced by gussets, based on the theory presented in Timoshenko and Gere [3]

In formula (8) it was assumed that factor η substitutes for $\frac{\mu}{GA}$. Factor η is a value that shear force V_p needs to be multiplied by to get an additional line inclination angle, resulting from shear action. Hence, we get:

$$\gamma = \eta \cdot V_p \quad (10)$$

In order to obtain angular displacement caused by shear force V_p, the strain of an element of the column, located between cross sections α–α and β–β, as in Fig. 1a, was considered. Assuming that deflection curves of the column shafts have their inflection points in those sections, the bending forms of the element concerned that are shown in Fig. 1b, c were derived. Cumulative angular displacement γ which is the result of the action of force V_p, consists of angular displacement caused by bending of the shaft and angular displacement resulting from bending and shearing of the gussets.

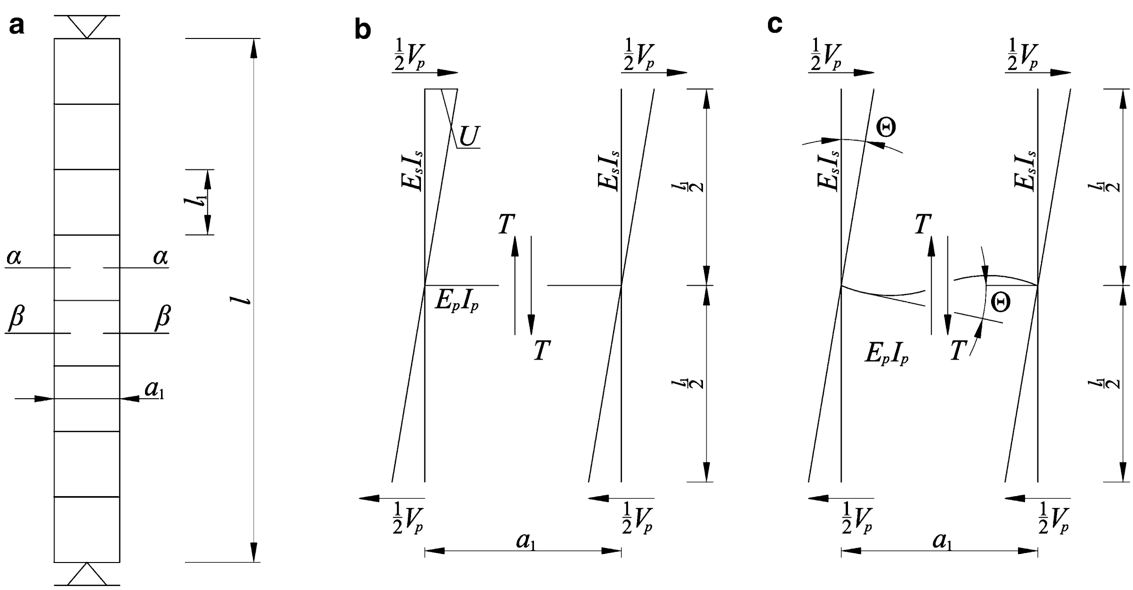

Fig. 1 Spaced column. **a** Static diagram of the column. **b** Deformed element of the column due to bending of its shaft. **c** Deformed element of the column due to bending and shearing of its gusset

Determination of angular displacement arising from shaft bending (Fig. 1b)

In order to determine angular displacement arising from shaft bending, horizontal displacements u, as per Fig. 1b, were determined.

$$u = \int_0^{\frac{l_1}{2}} \frac{\frac{V_p}{2} \cdot x \cdot \overline{1} \cdot x}{E_s I_s} dx = \frac{V_p \cdot l_1^3}{48 E_s I_s} \tag{11}$$

Then, angular displacement was drawn which equals $\frac{u}{0.5 \cdot l_1}$.

The calculations performed are based on the Maxwell–Mohr formula.

Determination of angular displacement arising from bending of gussets (Fig. 1c)

Angular displacement Θ, resulting from gusset bending, as in Fig. 1c amounts to:

$$\Theta = \int_0^{a_1} \frac{\left(\frac{V_p}{2} \cdot l_1 - \frac{V_p l_1}{a_1} \cdot x\right)\left(1 - \frac{x}{a_1}\right)}{2 E_p I_p} dx = \frac{V_p \cdot l_1 \cdot a_1}{24 E_p I_p} \tag{12}$$

The calculations performed are based on the Maxwell–Mohr formula.

Determination of angular displacement arising from shearing of the gussets (Fig. 1c)

Angular displacement arising from shearing of the gussets is $\frac{\mu}{2GA} T$.

Therefore, the total angular displacement γ is equal:

$$\gamma = \frac{u}{0.5 \cdot l_1} + \Theta + \frac{\mu}{2GA} T \tag{13}$$

Having replaced $u = \frac{V_p \cdot l_1^3}{48 E_s I_s}$, $\Theta = \frac{V_p \cdot l_1 \cdot a_1}{24 E_p I_p}$ and $T = \frac{V_p \cdot l_1}{a_1}$ in that Eq. (13), the following was obtained:

$$\gamma = \frac{V_p \cdot l_1^2}{24 E_s I_s} \psi + \frac{V_p \cdot l_1 \cdot a_1}{24 E_p I_p} + \frac{\mu \cdot V_p \cdot l_1}{2 G_p A_p \cdot a_1} \tag{14}$$

Buckling of the column shaft, between two neighboring gussets was taken into account by multiplying the first factor in Eq. (14) by the expression:

$$\psi = \frac{1}{1 - \frac{P_{\text{c,crit}}}{2 P_{\text{e,1}}}} \tag{15}$$

where $P_{e,1} = \frac{\pi^2 E_s I_s}{l_1^2}$ is the critical force active in buckling of a column shaft between its gussets. Using equations (10) and (14), the value of η shall be:

$$\eta = \frac{l_1^2}{24 E_s I_s} \psi + \frac{l_1 \cdot a_1}{24 E_p I_p} + \frac{\mu \cdot l_1}{2 G_p A_p \cdot a_1} \tag{16}$$

By replacing expression (16) in formula (8) instead of $\frac{\mu}{GA}$ and substituting mean moduli E_s, E_p, G_p with fifth percentiles $E_{0.05}^s$, $E_{0.05}^p$, $G_{0.05}^p$, the following formula for critical force was derived $P_{c,\text{crit}}$

$$P_{c,\text{crit}} = \frac{P_e}{1 + P_e\left(\frac{l_1^2}{24 E_{0.05}^S I_S} \psi + \frac{l_1 \cdot a_1}{24 E_{0.05}^p I_p} + \frac{\mu \cdot l_1}{2 G_{0.05}^p A_p \cdot a_1}\right)} \tag{17}$$

where $P_e = \frac{\pi^2 E_{0.05} I}{l_c^2}$ is the critical force for a column treated as a solid one. Due to the fact that in the denominator of formula (17) there is coefficient ψ derived from formula (15), dependent on critical force $P_{c,\text{crit}}$ that is sought for, Timoshenko and Gere [3] approach the problem of determining force $P_{c,\text{crit}}$ based on formula (17) as a non-linear one. As in such circumstances it is impossible to derive $P_{c,\text{crit}}$ directly from the formula, Timoshenko and Gere [3] state that Eq. (17) can be solved only by the trial method. As the trial method is complicated and consists merely of numeric calculations, the author solved the problem by creating a formula, from which critical force $P_{c,\text{crit}}$ can be derived directly and in a precise manner. Performing a number of transformations, the author brought the formula (17) down to the form of a quadratic trinomial, where the requested variable is critical force $P_{c,\text{crit}}$ itself. This way the following equation was derived:

$$\begin{aligned} &\left[\frac{\lambda^2 \lambda_1^2}{\pi^2 E_{0.05} A} + \lambda_1^2 (\eta - \eta_1)\right] P_{c,\text{crit}}^2 \\ &- \left[\lambda^2 + \pi^2 E_{0.05} A \cdot \eta + \lambda_1^2\right] P_{c,\text{crit}} + \pi^2 E_{0.05} A = 0 \end{aligned} \tag{18}$$

Then, the author solved Eq. (18), deriving $P_{c,\text{crit}}$ function from it:

$$P_{c,\text{crit}} = \frac{\left[\lambda^2 + \pi^2 E_{0.05} \cdot A \cdot \eta + \lambda_1^2\right] - \sqrt{\left[\lambda^2 + \pi^2 E_{0.05} \cdot A \cdot \eta + \lambda_1^2\right]^2 - 4\left[\frac{\lambda^2 \cdot \lambda_1^2}{\pi^2 E_{0.05} \cdot A} + \lambda_1^2 (\eta - \eta_1)\right] \pi^2 E_{0.05} \cdot A}}{2\left[\frac{\lambda^2 \cdot \lambda_1^2}{\pi^2 E_{0.05} \cdot A} + \lambda_1^2 (\eta - \eta_1)\right]} \tag{19}$$

Therefore, basing on formula (17), the buckling length of a spaced column is the following:

$$l_{ef}=\sqrt{1+P_e\left(\frac{l_1^2}{24E^s_{0.05}I_s}\psi+\frac{l_1a_1}{24E^p_{0.05}I_p}+\frac{\mu l_1}{2G^p_{0.05}A_pa_1}\right)}\cdot l_c \tag{20}$$

Hence, the author suggests application of the following formula for column slenderness ratio λ_{ef}. The formula was derived on the basis of dependencies (19), (20), and it has the form:

$$\lambda_{ef}=\sqrt{\lambda^2+\pi^2E_{0.05}\cdot A\left(\eta_1\cdot\psi+\eta_2+\eta_3\right)} \tag{21}$$

based on dependence (15) $\psi=\dfrac{1}{1-\dfrac{P_{c,\mathrm{crit}}}{2P_{e,1}}}$, where $P_{e,1}=\dfrac{\pi^2E^s_{0.05}I_s}{l_1^2}$ and formula (19), coefficient ψ is expressed with the formula:

$$\psi=\frac{2\lambda^2+2\pi^2E_{0.05}\cdot A\left(\eta-\eta_1\right)}{\lambda^2+\pi^2E_{0.05}A\left(\eta-2\eta_1\right)-\lambda_1^2+\sqrt{\left[\lambda^2+\pi^2E_{0.05}A\cdot\eta+\lambda_1^2\right]^2-4\left[\lambda^2+\pi^2E_{0.05}A\left(\eta-\eta_1\right)\right]\lambda_1^2}} \tag{22}$$

$$\eta_1=\frac{l_1^2}{24E^s_{0.05}I_s} \tag{23}$$

$$\eta_2=\frac{l_1\cdot a_1}{24E^p_{0.05}I_p} \tag{24}$$

$$\eta_3=\frac{\mu\cdot l_1}{2G^p_{0.05}A_pa_1} \tag{25}$$

$$\eta=\eta_1+\eta_2+\eta_3 \tag{26}$$

$$\lambda=\frac{l_c}{i} \tag{27}$$

$$\lambda_1=\frac{l_1}{i_1} \tag{28}$$

Calculation of gussets based on the theory presented in Timoshenko and Gere [3]

When designing the columns discussed herein, correct design of gussets is of critical meaning. To calculate gussets we need to rely on the fact that the theoretical model in question has some geometrical imperfections, such as initial curvature or load eccentricity. In order to determine shear forces, the differential equation of a deformed axis of a column is used where its initial curvature is described with a sinusoid and forces P act upon eccentric e. The author analyzed two least favorable - according to Timoshenko and Gere [3] - static schemes of the column, as shown in Fig. 2. Figure 2a shows a buckled shape of a column at its initial curvature described with a sinusoid. Figure 2b presents a buckled shape of a column at its load acting on eccentrics e, where the eccentrics are of contradictory orientation.

Case 1: determination of shear forces V_p and $V_{p,\max}$ based on the analysis of a column at its initial curvature described with a sinusoid

Determination of maximum shear force V_{p} caused by longitudinal compressive force P

Upon determining shear forces V_p, the differential equation of a deformed axis of a column:

$$\frac{d^2y(x)}{dx^2}+k^2y(x)=-\frac{M(x)}{EI} \tag{29}$$

where $k^2=\dfrac{P}{EI}$ as well as the formula for normal compressive stress in a column eccentrically compressed were used.

$$\sigma_{c,0,k}=\frac{P}{A}+\frac{M_{\max}\cdot z_{\max}}{I}\le f_{c,0,k} \tag{30}$$

The differential equation of a deformed axis of a column at its initial curvature described with a sinusoid, based on Eq. (29), has the following form:

$$\frac{d^2y_{\square}(x)}{dx^2}+k^2y_{\square}(x)=-k^2a\cdot\sin\frac{\pi}{l}x \tag{31}$$

This equation was solved with the operational calculus, based on the Laplace transformation Osiowski [6], and the solution for that is the following function:

$$y_{\square}(x)=w_0\cos kx+w_1\frac{1}{k}\sin kx+\frac{k^2\cdot l^2}{\pi^2-k^2\cdot l^2}a\sin\frac{\pi}{l}x-\frac{k^2\cdot l^2}{\pi^2-k^2\cdot l^2}a\sin kx \tag{32}$$

Having taken the boundary conditions into consideration and added initial curvature $y_p(x)=a\sin\frac{\pi}{l}x$, the following equation was derived:

$$y(x)=\frac{a}{1-\alpha}\sin\frac{\pi}{l}x \tag{33}$$

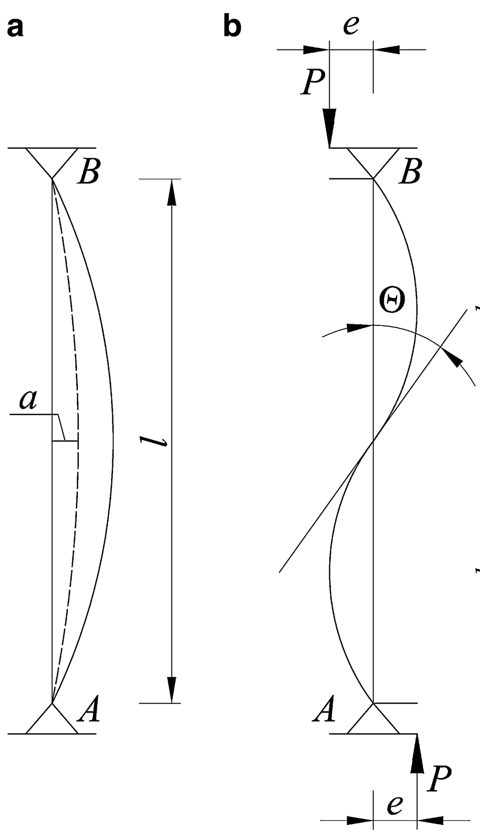

Fig. 2 Buckled forms of the column. **a** Buckled column, with its initial curvature described with a sinusoid. **b** Buckled column, with forces P acting on eccentric e

$\alpha = \frac{P}{P_e}$, where P_e is identified by the formula $P_e = \frac{\pi^2 EI}{l_c^2}$. In order to determine shear force, depending on compressive force P, angular displacement of a column on a support was determined:

$$\Theta = \left[\frac{dy(x)}{dx}\right]_{x=0} = \frac{\pi \cdot a}{l(1-\alpha)} \tag{34}$$

hence, for the column length of $l = l_{\mathrm{ef}}$, shear force V_p is:

$$V_p = P\frac{\pi \cdot a}{l_{ef}(1-\alpha)} \tag{35}$$

and when transformed, it is:

$$V_p = P\frac{a}{i}\frac{\pi^3 E_{0.05} A}{\pi^2 E_{0.05} A \lambda_{ef} - P\lambda_{ef}^3} \tag{36}$$

Determination of maximum shear force $V_{p,\max}$ the column can carry

In order to determine shear force $V_{p,\max}$ the given column can carry, we divide formula (35) on both sides by column cross section A. Using $\sigma_{\mathrm{mid}} = \frac{P}{A}$, to identify mean compressive stress, at which yields an extreme grain of the column cross section begins to develop, we find that the value of the highest shear force acting on a unit of the cross section, at which the yield begins, is:

$$\frac{V_{p,\max}}{A} = \sigma_{\mathrm{mid}}\frac{\pi a}{l_{ef}(1-\alpha)} \tag{37}$$

Equation (37) can be transformed, using the formula (30). Assuming therein $\sigma_{\mathrm{mid}} = \frac{P}{A}$ and $M_{\max} = P \cdot y_{\max}$, where $y_{\max}$, based on Eq. (33), is $y_{\max} = \frac{a}{1-\alpha}$, we receive:

$$\sigma_{\max} = \sigma_{\mathrm{mid}}\left[1 + \frac{a}{c}\frac{1}{1-\alpha}\right] = f_{c,0,k} \tag{38}$$

where $c = \frac{i^2}{z_{\max}}$.

This dependence identifies maximum compressive stress, occurring in extreme grain, equal to the characteristic compressive strength of wood caused by longitudinal compressive force P, acting on eccentric $y_{\max}$ resulting from Eq. (33). Deriving value of a from Eq. (38) and replacing it in Eq. (37), we get:

$$\frac{V_{p,\max}}{A} = \frac{\pi \cdot c}{l_{ef}}\left[f_{c,0,k} - \sigma_{\mathrm{mid}}\right] \tag{39}$$

Stress σ_{mid} was derived in the following way. Equation (38) was used. Dividing both numerator and denominator of expression α by cross section A, after some transformations, $\alpha = \frac{\sigma_{\mathrm{mid}}}{\sigma_{c,\mathrm{crit}}}$ was obtained, where $\sigma_{\mathrm{mid}} = \frac{P}{A}$ $\sigma_{c,\mathrm{crit}} = \frac{\pi^2 E_{0.05}}{\lambda^2}$ $\lambda = \frac{l_c}{i}$. Having solved Eq. (38) towards σ_{mid}, the following quadratic equation appeared:

$$\sigma_{\mathrm{mid}}^2 - \left[f_{c,0,k} + \left(1 + \frac{a}{c}\right)\sigma_{c,\mathrm{crit}}\right]\sigma_{\mathrm{mid}} + f_{c,0,k} \cdot \sigma_{c,\mathrm{crit}} = 0 \tag{40}$$

This equation has two real roots $\sigma_{\mathrm{mid}\,(1)}$ and $\sigma_{\mathrm{mid}\,(2)}$:

$$\sigma_{\mathrm{mid}(1)(2)} = \frac{1}{2}\left[f_{c,0,k} + \left(1 + \frac{a}{c}\right)\sigma_{c,\mathrm{crit}} \mp \sqrt{\left[f_{c,0,k} + \left(1 + \frac{a}{c}\right)\sigma_{c,\mathrm{crit}}\right]^2 - 4f_{c,0,k}\sigma_{c,\mathrm{crit}}}\right] \tag{41}$$

Taking into account the smaller root $\sigma_{\mathrm{mid}} = \sigma_{\mathrm{mid}(1)}$ and putting it in Eq. (39), after some transformation, the following equation was derived:

$$\frac{V_{p,\max}}{A} = \frac{\pi c}{2 l_{ef}}\left[f_{c,0,k} - \left(1 + \frac{a}{c}\right)\frac{\pi^2 E_{0.05}}{\lambda_{ef}^2} + \sqrt{\left[f_{c,0,k} + \left(1 + \frac{a}{c}\right)\frac{\pi^2 E_{0.05}}{\lambda_{ef}^2}\right]^2 - 4f_{c,0,k}\frac{\pi^2 E_{0.05}}{\lambda_{ef}^2}}\right] \tag{42}$$

Case 2: determination of shear forces V_p and $V_{p,\max}$, based on the analysis of a column at forces P acting on eccentric e

Determination of maximum shear force V_p, caused by longitudinal compressive force P

Based on Eq. (29), a differential equation of a deformed axis of a column at compressive force P, acting on support B, was formulated as per Fig. 2b. The equation has the following form:

$$\frac{d^2y(x)}{dx^2} + k^2y(x) = -\frac{1}{EI}\frac{P \cdot e}{l}x \tag{43}$$

where $k^2 = \frac{P}{EI}$.

This equation was solved with the operational calculus, based on the Laplace transformation Osiowski [6], and the solution for that is the following function:

$$y(x) = w_0 \cos kx + w_1 \frac{1}{k} \sin kx - \frac{e}{l}x + \frac{e}{l \cdot k} \sin kx \tag{44}$$

Having taken the boundary conditions into consideration this function was formulated:

$$y(x) = e\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right) \tag{45}$$

Next, the column central cross section angular displacement Θ was determined, treating each half of the column as a compressed rod of length $\frac{l}{2}$, simply supported and loaded with bending moment $P{\cdot}e$ as shown in Fig. 2b. By differentiating Eq. (45) and replacing l with $\frac{l}{2}$, the following was obtained:

$$\Theta = \left[\frac{dy(x)}{dx}\right]_{x=0} = \frac{e}{l}\left[\frac{kl}{\sin\frac{kl}{2}} - 2\right] \tag{46}$$

Depending on compressive force P at $l = l_{ef}$ is:

$$\begin{aligned} V_p &= \frac{2P \cdot e}{l_{ef}} + P \cdot \Theta = \frac{2P \cdot e}{l_{ef}} + \frac{P \cdot e}{l_{ef}}\left(\frac{kl_{ef}}{\sin\frac{kl_{ef}}{2}} - 2\right) \\ &= P \cdot e \frac{k}{\sin\frac{kl_{ef}}{2}} \end{aligned} \tag{47}$$

and after some transformations:

$$V_p = P\frac{e}{i}\frac{\left(\frac{P}{E_{0.05}A}\right)^{0.5}}{\sin\left[\frac{\lambda_{ef}}{2}\left(\frac{P}{E_{0.05}A}\right)^{0.5}\right]} \tag{48}$$

Determination of maximum shear force $V_{p,\max}$ the column can carry

In order to determine maximum shear force $V_{p,\max}$ for a given column, we divide formula (47) on both sides by cross section A. Using $\sigma_{\text{mid}} = \frac{P}{A}$ to identify mean compressive stress, at which yields an extreme grain of the column cross section begins to develop, we find that the value of the highest shear force acting on a unit of the cross section, at which the yield begins, is:

$$\frac{V_{p,\max}}{A} = \sigma_{\text{mid}}\frac{e}{i}\frac{\left(\frac{\sigma_{\text{mid}}}{E_{0.05}}\right)^{0.5}}{\sin\left[\frac{\lambda_{ef}}{2}\left(\frac{\sigma_{\text{mid}}}{E_{0.05}}\right)^{0.5}\right]} \tag{49}$$

value σ_{mid} was established in the following way. Based on Eq. (29) a differential equation of a deformed axis of a column at compressive force P, acting on support A, was formulated as per Fig. 2b. The equation has the following form

$$\frac{d^2y(x)}{dx^2} + k^2y(x) = \frac{1}{EI}\frac{P \cdot e}{l}(l - x) \tag{50}$$

where $k^2 = \frac{P}{EI}$.

This equation was solved with the operational calculus, based on the Laplace transformation Osiowski [6], and the solution for that is the following function:

$$\begin{aligned} y(x) &= w_0 \cos kx + w_1 \frac{1}{k} \sin kx + \frac{e}{l}(l - x) - \frac{e}{l} \cos kx \\ &\quad + \frac{e}{l \cdot k} \sin kx \end{aligned} \tag{51}$$

Having taken the boundary conditions into consideration, this function appears as the solution:

$$y(x) = -e\left[\frac{\sin k(l - x)}{\sin kl} - \frac{l - x}{l}\right] \tag{52}$$

based on equations (45) and (52), total buckling from both forces P acting on eccentrics e is as follows:

$$y(x) = e\left[\frac{\sin kx}{\sin kl} - \frac{x}{l}\right] - e\left[\frac{\sin k(l - x)}{\sin kl} - \frac{l - x}{l}\right] \tag{53}$$

after some transformations, $y(x)$ equals:

$$y(x) = e\frac{\sin k\left(x - \frac{l}{2}\right)}{\sin\frac{kl}{2}} + e\left(1 - 2\frac{x}{l}\right) \tag{54}$$

Based on formula (54), maximum bending moment amounts to:

$$M_{\max} = -EI\left[\frac{d^2y(x)}{dx^2}\right]_{x=\left(\frac{\pi}{2 \cdot k} + \frac{l}{2}\right)} = P \cdot e \cdot \operatorname{cosec}\frac{kl}{2} \tag{55}$$

Having included the dependence (55) in formula (30) and assuming there that $\sigma_{\text{mid}} = \frac{P}{A}$ and $k = \sqrt{\frac{P}{EI}}$, after some transformations, we obtain:

$$\sigma_{\text{mid}}\left\{1 + \frac{e}{c}\operatorname{cosec}\left[\frac{\lambda_{ef}}{2}\left(\frac{\sigma_{\text{mid}}}{E_{0.05}}\right)^{0.5}\right]\right\} = f_{c,0,k} \tag{56}$$

In conclusion, the formulae, from which we derive maximum shear forces V_p acting in the columns discussed herein, have the following form:

$$V_p = \max \begin{cases} P\dfrac{a}{i}\dfrac{\pi^3 E_{0.05} A}{\pi^2 E_{0.05} A \cdot \lambda_{ef} - P\lambda_{ef}^3} & (57) \\ P \cdot \dfrac{e}{i}\dfrac{\left(\dfrac{P}{E_{0.05} A}\right)^{0.5}}{\sin\left[\dfrac{\lambda_{ef}}{2}\left(\dfrac{P}{E_{0.05} A}\right)^{0.5}\right]} & (58) \end{cases}$$

The formulae, from which we derive maximum shear forces $V_{p,\max}$ the given column can carry, have the following form:

$$V_{p,\max} = \min \begin{cases} \dfrac{1}{2}\dfrac{c}{i}\dfrac{\pi A}{\lambda_{ef}}\left[f_{c,0,k} - \left(1+\dfrac{a}{c}\right)\dfrac{\pi^2 E_{0.05}}{\lambda_{ef}^2} + \sqrt{\left[f_{c,0,k} + \left(1+\dfrac{a}{c}\right)\dfrac{\pi^2 E_{0.05}}{\lambda_{ef}^2}\right]^2 - 4 f_{c,0,k}\dfrac{\pi^2 E_{0.05}}{\lambda_{ef}^2}}\right] & (59) \\ \sigma_{\text{mid}} \cdot \dfrac{eA}{i}\dfrac{\left(\dfrac{\sigma_{\text{mid}}}{E_{0.05}}\right)^{0.5}}{\sin\left[\dfrac{\lambda_{ef}}{2}\left(\dfrac{\sigma_{\text{mid}}}{E_{0.05}}\right)^{0.5}\right]} & (60) \end{cases}$$

mean stress σ_{mid} is derived from this equation:

$$\sigma_{\text{mid}}\left\{1 + \frac{e}{c}\operatorname{cosec}\left[\frac{\lambda_{ef}}{2}\left(\frac{\sigma_{\text{mid}}}{E_{0.05}}\right)^{0.5}\right]\right\} - f_{c,0,k} = 0 \qquad (61)$$

Determination of initial maximum curvature of column *a*, and of force application eccentric *e*

In accordance with the literature (EN-1995 Eurocode 5 [4]), upon determining instability factor k_c, based on column buckling line at its initial curvature described with a sinusoid, expression $\frac{a}{c}$ was substituted with $\beta\left(\frac{\lambda_{ef}}{\pi}\sqrt{\frac{f_{c,0,k}}{E_{0.05}}} - 0.3\right)$, where coefficient β was $\beta = 0.2$ for solid timber and $\beta = 0.1$ for glued laminated timber. Therefore, the author suggests using the following formulae, concerning a and e.

$$a = e = \begin{cases} \dfrac{i^2}{z_{\max}}\left[\dfrac{\lambda_{ef}}{5\pi}\left(\dfrac{f_{c,0,k}}{E_{0.05}}\right)^{0.5} - 0.06\right] & \text{for solid timber} \quad (62) \\ \dfrac{i^2}{z_{\max}}\left[\dfrac{\lambda_{ef}}{10\pi}\left(\dfrac{f_{c,0,k}}{E_{0.05}}\right)^{0.5} - 0.03\right] & \text{for glued laminated timber} \quad (63) \end{cases}$$

Comparative static analysis of load-bearing capacity of columns

A comparative static analysis of columns was conducted in regards to their load-bearing capacity, using the formulae given in EN-1995 Eurocode 5 [4] and those based on the theory presented in Timoshenko and Gere [3]. The comparative analysis was based on a value λ_{ef} described with the formulae (1) and (21). The formula (1) was used to determine critical load-bearing capacity of a column without taking the influence of shearing into consideration. However, the formula (21) was used to determine critical load-bearing capacity taking the influence of shearing into consideration.

Table 1 shows strength, mean values of both moduli of elasticity and of shear of materials used for the analysis of the columns.

Fifth percentiles of moduli of elasticity and of shear were drawn from formulae $E_{0.05} \approx \frac{E_{\text{mean}}}{1.5}, G_{0.05} \approx \frac{G_{\text{mean}}}{1.5}$. Load-bearing capacities were derived from the formula $n = \frac{P}{A \cdot k_c \cdot f_{c,0,d}}$. In the analysis, columns of three slenderness ratio values $\lambda_{ef} = 50$, maximum slenderness ratio $\lambda_{ef} = 150$ and intermediate slenderness ratio $\lambda_{ef} = 100$ were discussed. The slenderness ratios $\lambda_{ef} = [50, 100, 150]$ refer to calculations based on the formulae given in EN-1995 Eurocode 5 [4]. For the analysis, columns were a shaft size of 80·80 mm were adopted. The space between them was assumed at $a = 60$ mm. It was also assumed that the gussets in the columns were spaced axially at $l_1 = 600$ mm. It was assumed that shafts of the columns would be solid timber

Table 1 Static values used in the analysis of columns. Standards EN 338 [1], EN 12369-2 [7], EN 12369-1 [8]

	$f_{r,k}$ (MPa)	$f_{v,k}$ (MPa)	$f_{m,k}$ (MPa)	E_{mean} (MPa)	G_{mean} (MPa)
Timber C18	3.4	3.4	18	9000	560
Plywood	1.2	7.5	30	6000	550
Chipboard	1.8	6.6	14.2	3200	860
Fibre board	3.0	18	35	4800	2000

In Table 1:

$f_{r,k}$ – characteristic planar (rolling) shear strength

$f_{v,k}$ – characteristic panel shear strength

$f_{m,k}$ – characteristic bending strength

E_{mean} – mean value of modulus of elasticity

G_{mean} – mean value of shear modulus, mean modulus of rigidity

C18, and that their gussets would be produced, as variants, from timber, plywood, particleboard and fibreboard. It was assumed that the solid timber gussets would be made from timber class C18. It was assumed that the gussets were attached to the column shaft with glue. Aminoplastic resin, phenolic resin and polycondensation adhesive, described in the standard EN 301 [9], are among the possible materials that can be used for glueing. The height of the gussets was chosen appropriately to fulfil the load-bearing capacity condition for a glued joint, connecting the gusset with the column shaft, affected by torsional moment and shearing force. Then, the gusset thicknesses were chosen in such a way that they would be able to carry the shearing force and the bending moment created by it. Values of longitudinal compressive forces in the columns were chosen so that the load-bearing capacities of the columns, calculated with the formulae given in EN-1995 Eurocode 5 [4], accounting for the assumed dimensions of the column elements, were $n=1.0$. Load-bearing capacity $n=1.0$ means that stresses taking place in a stressed element are of equal value to permissible stress values for the given class of timber. Values $n>1$ indicate by how much the load-bearing capacity has been exceeded. Values $n<1$ indicate the existing reserve in the load-bearing capacity. In order to determine how much the load-bearing capacity has been exceeded and what reserve still exists in it, one has to calculate the value of $(n-1)\cdot 100\%$.

Tables 2, 3 and 4 present some static values determined based on the comparative analysis. These values include slenderness ratios λ_{ef}, instability factors k_c, load-bearing capacity n and percentage differences $(n-1)\cdot 100\%$. The values were determined on the basis of the formulae given in EN-1995 Eurocode 5 [4] and those based on the theory presented in Timoshenko and Gere [3].

Table 2 Analysis of load-bearing capacity of columns

	$\lambda_{ef}=50$				
	1	2	3	4	5
λ_{ef}	50	52.07	72.75	60.63	67.50
k_c	0.781	0.756	0.500	0.647	0.560
n	1.0	1.033	1.560	1.208	1.394
$(n-1)\cdot 100\%$	0%	3.3%	56%	20.8%	39.4%

Table 3 Analysis of load-bearing capacity of columns

	$\lambda_{ef}=100$				
	1	2	3	4	5
λ_{ef}	100	99.08	110.76	103.37	107.01
k_c	0.290	0.295	0.240	0.273	0.256
n	1.0	0.984	1.207	1.062	1.132
$(n-1)\cdot 100\%$	0%	−1.6%	20.7%	6.2%	13.2%

Table 4 Analysis of load-bearing capacity of columns

	$\lambda_{ef}=150$				
	1	2	3	4	5
λ_{ef}	150	149.31	157.32	152.19	154.70
k_c	0.135	0.137	0.124	0.132	0.128
n	1.0	0.988	1.092	1.024	1.057
$(n-1)\cdot 100\%$	0%	−1.2%	9.2%	2.4%	5.7%

In Tables 2, 3 and 4:

– Item 1: calculations of columns, based on the formulae applied as per EN-1995 Eurocode 5 [4]

– Items 2, 3, 4, 5: calculations with the formulae based on the theory presented in Timoshenko and Gere [3], where:

2: Calculations of columns with timber gussets

3: Calculations of columns with plywood gussets

4: Calculations of columns with chipboard gussets

5: Calculations of columns with fibreboard gussets

λ_{ef} – as per formulae (1) and (21)

k_c – instability factor

n – load-bearing capacity of a column

Comparative static analysis for calculation of gussets

Within the framework of this study, a comparative static analysis of maximum shear forces V_p, caused by longitudinal force P, and maximum shear forces $V_{p,\max}$ a column can carry, was conducted. The comparative analysis was based on the formulae (2) and (57), (58), (59) and (60). The formula (2) was used to determine shear forces in the column basing on the used to date theory given in literature (EN-1995 Eurocode 5 [4]) which does not take into consideration the influence of shearing on critical load-bearing capacity. Formulas (57), (58), (59) and (60) were used to determine shear forces in the column basing on the theory given in Timoshenko and Gere [3], which takes into consideration the influence of shearing on critical load-bearing capacity.

Elastic and strength values of the materials in use are presented in Table 1. In the analysis, columns with three slenderness ratios: $\lambda_{ef}=50$, $\lambda_{ef}=100$ and $\lambda_{ef}=150$ were taken into account; the shaft discussed herein, with slenderness ratio $\lambda_{ef}=150$, was assumed, where the calculations were performed for variants of columns built with timber, plywood, particleboard or fibreboard gussets. In all cases, one longitudinal compressive force P, causing maximum normal stress in the shaft discussed herein, with slenderness ratio $\lambda_{ef}=150$, was assumed, where the calculations were based on the formulae given in EN-1995 Eurocode 5 [4]. Values of maximum initial curvature of a column a and force eccentric e were derived from formulae (62) and (63).

Results of the analysis are presented in Tables 5, 6 and 7.

Table 5 Analysis of gussets

$\lambda_{ef}=50$					
	1	2	3	4	5
V_p^a	0.38	0.11	0.14	0.13	0.14
V_p^e		0.07	0.08	0.08	0.08
$V_{p,\max}^a$	2.22	2.27	3.33	2.82	3.16
$V_{p,\max}^e$		0.88	1.28	1.03	1.17

Table 6 Analysis of gussets

$\lambda_{ef}=100$					
	1	2	3	4	5
V_p^a	1.24	0.18	0.21	0.19	0.20
V_p^e		0.09	0.10	0.10	0.10
$V_{p,\max}^a$	2.66	3.45	3.33	3.41	3.37
$V_{p,\max}^e$		2.00	2.34	2.12	2.23

Table 7 Analysis of gussets

$\lambda_{ef}=150$					
	1	2	3	4	5
V_p^a	2.66	0.38	0.47	0.41	0.44
V_p^e		0.12	0.12	0.12	0.12
$V_{p,\max}^a$	2.66	2.81	2.70	2.77	2.74
$V_{p,\max}^e$		3.11	3.18	3.14	3.16

In Tables 5, 6 and 7:

- Item 1: calculations of columns, based on the formulae applied as per EN-1995 Eurocode 5 [4]
- Items 2, 3, 4, 5: calculations with the formulae based on the theory presented in Timoshenko and Gere [3], where:

  2: Calculations of columns with timber gussets

  3: Calculations of columns with plywood gussets

  4: Calculations of columns with chipboard gussets

  5: Calculations of columns with fiberboard gussets

  V_p^a – maximum shear force, caused by compressive force P, for a column with its initial curvature described with a sinusoid

  V_p^e – maximum shear force, caused by compressive force P, for a column, where force P acts on eccentric e

  $V_{p,\max}^a$ – maximum shear force a column can carry, for a column with its initial curvature described with a sinusoid

  $V_{p,\max}^e$ – maximum shear force a column can carry, for a column, where force P acts on eccentric e

Experimental studies

The columns load-bearing capacity formulae, derived in the paper, can be verified experimentally by conducting comparative analysis for instability factors k_c and load-bearing capacity n. A description of non-destructive tests concerning instability factor k_c is given in the author's article Śliwka [5], while the destructive tests for load-bearing capacity consist in the determination of forces destroying an element under study, and comparing them with forces established through theoretical analysis.

Summary and conclusions

1. The comparative static analysis of columns presented herein leads to the conclusion that the column calculation method set forth in EN-1995 Eurocode 5 [4] and the method based on the theory presented in Timoshenko and Gere [3] lead to partially divergent results.
2. The analysis concerning load-bearing capacity of the columns suggests that the smallest differences among the compared static values λ_{ef}, k_c, n, $(n-1)\cdot 100\%$ are observed in columns with slenderness ratio $\lambda_{ef}=150$ with timber gussets, while the biggest differences occur

in columns with slenderness ratio $\lambda_{ef}=50$ with gussets made of wood-based materials, especially of plywood.

2.1 For columns with slenderness ratio $\lambda_{ef}=150$ with timber gussets, slenderness ratio determined with the formulae based on the theory presented in Timoshenko and Gere [3] is 0.46% lower than slenderness ratio determined with calculations based on the formulae given in EN-1995 Eurocode 5 [4]. Instability factor k_c determined with the formulae based on the theory presented in Timoshenko and Gere [3] is 1.48% higher than instability factor determined with calculations based on the formulae given in EN-1995 Eurocode 5 [4]. Load-bearing capacity determined with the formulae based on the theory presented in Timoshenko and Gere [3] is unutilized in 1.25%, comparing to load-bearing capacity $n=1$ determined with the formulae given in EN-1995 Eurocode 5 [4]. Tables 2, 3, 4.

2.2 For columns with slenderness ratio $\lambda_{ef}=50$ with plywood gussets, slenderness ratio determined with the formulae based on the theory presented in Timoshenko and Gere [3] is 45.5% higher than slenderness ratio determined with calculations based on the formulae given in EN-1995 Eurocode 5 [4]. Instability factor k_c determined with the formulae based on the theory presented in Timoshenko and Gere [3] is 35.98% lower than instability factor determined with calculations based on the formulae given in EN-1995 Eurocode 5 [4]. Load-bearing capacity determined with the formulae based on the theory presented in Timoshenko and Gere [3] is exceeded by 56% in comparison with load-bearing capacity $n=1$, determined with calculations based on the formulae given in EN-1995 Eurocode 5 [4]. Tables 2, 3, 4.

3. As part of the comparative analysis concerning the calculation of the gussets, it was proven that shearing forces calculated with the formulae given in EN-1995 Eurocode 5 [4] were much higher than the shearing forces calculated with the formulae based on the theory presented in Timoshenko and Gere [3]. The smallest differences amongst the values of those forces are observed in the columns with slenderness ratios $\lambda_{ef}=50$ and plywood gussets, for a column with its initial curvature described with a sinusoid, while the biggest differences appeared in the columns with slenderness ratios $\lambda_{ef}=150$ with timber, plywood, particleboard, fibreboard, for a column with forces P acting on an eccentric e. Tables 5, 6, 7.
4. Shear forces calculated with the formulae based on the theory presented in Timoshenko and Gere [3] are stronger in the case of a column with its initial curvature described with a sinusoid than in the case, where forces P act on eccentric e.
5. Moreover, it was demonstrated that the highest shearing forces $V_{p,\max}$ the column can bear, calculated with the formulae based on the theory presented in Timoshenko and Gere [3], differ from the maximum shearing forces calculated with the formulae given in EN-1995 Eurocode 5 [4]. The smallest differences are observed in columns with slenderness ratio $\lambda_{ef}=150$ with plywood gussets, for a rod with its initial curvature described with a sinusoid, while the biggest differences were seen amongst columns with slenderness ratio $\lambda_{ef}=50$ with timber gussets, for a column with forces P acting on an eccentric e. The analysis shows that among the group of cases in question maximum shearing forces $V_{p,\max}$ that can be transferred onto a column, calculated basing on Timoshenko and Gere [3], are lower than those maximum forces determined with the formulae (2) presented in the literature (EN-1995 Eurocode 5 [4]). Tables 5, 6, 7.
6. Summing up, in many considered cases, there exist fundamental differences between the compared static values. Assuming that the presented theory, given in Timoshenko and Gere [3], is a correct and accurate one, it can be stated that calculating the columns with the help of formulae used to date and given in the literature (EN-1995 Eurocode 5 [4]) can lead to design errors. Therefore, the author suggests applying the formulae based on the theory given in Timoshenko and Gere [3] when calculating the columns. The formulae presented in the paper have a practical significance in analyses of timber constructions.
7. The article presents the more thorough theory in comparison with the theory used so far, thus, it can constitute the author's indisputable contribution to the development in the scope of the stability of timber constructions.
8. Shear forces in a column depend directly on the initial curvature of a column, described with sinusoid a and, alternatively, on force application eccentric e. Therefore, the author suggests the formulation of functions $a=f(\lambda_{ef})$, $e=f(\lambda_{ef})$, which depend on slenderness ratio λ_{ef} of the column in question, described with formulae (62) and (63).
9. A static analysis of lattice columns with N and V lattice configuration is provided for in a separate paper by the author.

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