J Wood Sci (2018) 64:16–22
DOI 10.1007/s10086-017-1676-6

ORIGINAL ARTICLE

Practical techniques for the vibration method with additional mass: bending vibration generated by tapping cross section

Yoshitaka Kubojima[1] · Satomi Sonoda[2] · Hideo Kato[1]

Received: 26 July 2017 / Accepted: 7 October 2017 / Published online: 13 November 2017
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Abstract This work examines the effect of a method for generating bending vibration on the accuracy of a non-destructive and simple estimation of weight, density, and Young's modulus through a vibration test without measuring specimen weight. The resonance frequencies with and without the concentrated mass generated by tapping the RT (radial tangential)-plane under the free-free condition were compared with those generated by the normal free-free bending vibration. The air-dried specimens and wet specimens in a drying process at 20 °C and 65% relative humidity were used and then their weight, density, and Young's modulus were estimated by the vibration test. The appropriate resonance frequency of the bending vibration could be obtained by tapping the RT-plane. Generating bending vibration by tapping the RT-plane is effective for the application of the vibration method with additional mass to a drying process.

Keywords Bending vibration · Cross section · Longitudinal vibration · Vibration method with additional mass

Introduction

Adding mass to a bar decreases the resonance frequencies of the bar due to longitudinal and bending vibrations. With this approach, the weight, density, and Young's modulus of a bar can be calculated without weighing it [1–6]. This testing method is referred to as the vibration method with additional mass in this study, and this method enables to simply obtain properties of each piled lumber and each beam of timber guardrails.

Several test conditions have been studied to apply the vibration method with additional mass to actual cases. The suitable mass ratio (additional mass/specimen) and the connection way between the additional mass and specimen were experimentally examined. The accuracy of the evaluated material property decreased as the mass ratio increased. The optimum mass ratio for estimation of the material property was approximately 2%. The mean value of estimation error of test with the wood screws connection under the most suitable mass ratio indicated −1%. When using the adhesive connection, the mean value was approximately −7% [7]. The effect of the crosser's position used for piled lumber on longitudinal vibration was investigated. Placing crossers at the nodal positions ensures that accurate density and Young's modulus values can be determined using the vibration method with additional mass without the influence of weight of the upper lumber [8]. The effect of moisture content on the estimation accuracy of the vibration method with additional mass was examined. The estimation accuracy was affected by an increase in the resonance frequency, caused by the drying process during the vibration tests. The estimation accuracy in the bending vibration test was higher than that in the longitudinal vibration test [9].

Hence, the bending vibration test is suitable for piled lumber from the aspect of estimation accuracy. However, generating the bending vibration for wood inside the piled lumber is difficult. By contrast, generating the longitudinal vibration for such wood is easy.

✉ Yoshitaka Kubojima
kubojima@ffpri.affrc.go.jp

[1] Forestry and Forest Products Research Institute, 1 Matsunosato, Tsukuba, Ibaraki 305-8687, Japan

[2] Toyama Prefectural Agricultural, Forestry and Fisheries Research Center, 4940 Kurokawa Shin, Imizu, Toyama 939-0311, Japan

When a plate is tapped, many vibration modes such as flexural, torsional, longitudinal and other vibrations appear. The resonance frequencies of the flexural vibration modes, those of the torsional vibration modes and longitudinal vibration modes can be identified easily: when a rectangular bar is tapped along its edge line, a frequency spectrum of complex vibrations of bending and twisting occurs. By contrast, when the bar is tapped along its center line, pure bending vibration occurs [10, 11].

We attempted generating the bending vibration by tapping a cross section of wood and examined whether or not the vibration method with additional mass can be accurately performed using the obtained resonance frequencies.

Vibration method with additional mass

In the case of a thin beam, the effects of shear deflection and rotary inertia involved in the bending vibrational deflection are negligible, and Euler–Bernoulli elementary theory of bending can be applied to the vibration.

The Young's modulus using the bending vibrations E of a rectangular bar with length l, is expressed as follows:

$$E = \left(\frac{l}{m_{\mathrm{n}}}\right)^2 \frac{\rho A}{I}\omega_{\mathrm{n}}^2 \tag{1}$$

where ρ, ω, A, and I are the density, angular frequency ($\omega = 2\pi f$, f resonance frequency), cross-sectional area, and the moment of inertia of the cross section, respectively. The value of m_{n} is explained below.

The frequency equation for the free-free bending vibration with concentrated mass M placed at $x = al$ (x distance along the bar, $0 \le a \le 1$, $a + b = 1$) of a rectangular bar (Fig. 1) is expressed as follows [12]:

$$\begin{aligned}&\left(\cos m_{\mathrm{n}}\cosh m_{\mathrm{n}} - 1\right) - \frac{1}{2}\mu m_{\mathrm{n}}\left\{\left(\cos am_{\mathrm{n}}\cosh am_{\mathrm{n}} + 1\right)\right.\\&\quad\left(\sin bm_{\mathrm{n}}\cosh bm_{\mathrm{n}} - \cos bm_{\mathrm{n}}\sinh bm_{\mathrm{n}}\right)\\&\quad+\left(\cos bm_{\mathrm{n}}\cosh bm_{\mathrm{n}} + 1\right)\\&\quad\left.\left(\sin am_{\mathrm{n}}\cosh am_{\mathrm{n}} - \cos am_{\mathrm{n}}\sinh am_{\mathrm{n}}\right)\right\} = 0\end{aligned} \tag{2}$$

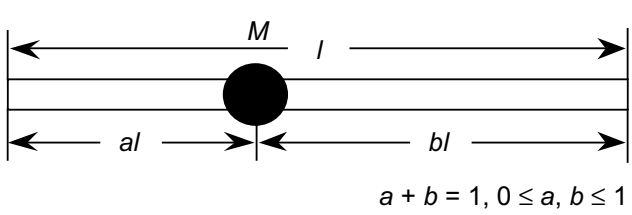

Fig. 1 A beam with additional mass

where μ is the ratio of the concentrated mass to the mass of the bar and is written as:

$$\mu = \frac{M}{\rho A l} \tag{3}$$

The suffix n is the resonance mode number.

If $\mu = 0$, Eq. (2) becomes

$$\cos m_{\mathrm{n0}}\cosh m_{\mathrm{n0}} - 1 = 0 \tag{4}$$

where the suffix 0 represents the value without the concentrated mass.

For a bar without a concentrated mass, Eq. (4) gives:

$$m_{10} = 4.730, \quad m_{20} = 7.853, \quad m_{30} = 10.996,$$
$$m_{\mathrm{n0}} = \frac{1}{2}(2\mathrm{n} + 1)\pi(\mathrm{n} > 3) \tag{5}$$

The density and Young's modulus are the same before and after the concentrated mass is bound to a specimen. Thus, using Eq. (1) obtains,

$$m_{\mathrm{n}} = \sqrt{\frac{f_{\mathrm{n}}}{f_{\mathrm{n0}}}}m_{\mathrm{n0}} \tag{6}$$

The value of μ can be calculated by substituting m_{n} from Eq. (6) into Eq. (2). The weight and density can be obtained by substituting the calculated μ, the concentrated mass, and the dimensions of a bar into Eq. (3). The Young's modulus can be calculated by substituting the density from Eq. (3) and the resonance frequency without the concentrated mass into Eq. (1) [1–6]. This procedure is referred to as the "vibration method with additional mass" in this study.

Materials and methods

Specimens

Sitka spruce (*Picea sitchensis* Carr.) was used as the sample specimen in this study. Air-dried specimens with dimensions of 1500–1600 mm (L longitudinal) × 100–105 mm (R radial) × 100–105 mm (T tangential) (large specimen) and those 300 mm (L) × 30 mm (R) × 5 mm (T) (small specimen) were made. Five large specimens and ten small specimens were used. The specimens were straight-grained and had no knots. The large specimens were conditioned at 20 °C and 65% relative humidity (RH). The tests were conducted under the same conditions.

Small specimens were placed under water and then conditioned at 20 °C and 65% RH until they attained a constant weight. The tests were performed directly after taking the specimen from water (0 h) and again after 3, 6, 9, 12, 15, 18, 21, and 24 h, and when the weight of the specimen became

constant at 20 °C and 65% RH (8 day). When the weight of the specimen became constant at 20 °C and 65% RH, the specimen is considered to be air-dried. After finishing the test under the air-dried condition, the small specimens were oven-dried at 105 °C.

The oven-dried weight was used to calculate the moisture content of the small specimen. The moisture content at 0 h varied from 104 to 128%.

The following nine steps shown in Table 1 were applied to the small specimens. All tests were conducted under 20 °C and 65% RH.

Step 1 The dimensions and weight of the small specimen were measured.

Step 2 The first free-free bending vibration test without the concentrated mass was performed. The vibration was generated by tapping the LR-plane of a specimen.

Step 3 The first vibration test by tapping the RT-plane of a specimen without the concentrated mass was performed.

Step 4 The specimen and the concentrated mass were weighed.

Step 5 The vibration test with the concentrated mass was performed. The vibration was generated by tapping the RT-plane of a specimen.

Step 6 The specimen was weighed.

Step 7 The second vibration test by tapping the RT-plane of a specimen without the concentrated mass was performed.

Step 8 The second free-free bending vibration test without the concentrated mass was performed. The vibration was generated by tapping the LR-plane of a specimen.

Step 9 The specimen was weighed.

The resonance frequency will increase with the decrease in the moisture content caused by drying during vibration tests, and the increase in the resonance frequency will affect the accuracy of the vibration method with additional mass. Hence, the first and second vibration tests without the concentrated mass (steps 2, 3, 7, and 8) were conducted [9].

Generation of longitudinal and bending vibrations by tapping a cross section

Longitudinal and bending vibrations were generated by tapping a cross section on the specimen with and without the concentrated mass using the following procedure as shown in Fig. 2. The test bar was suspended by two threads at the nodal positions of free-free vibration corresponding to its first resonance mode (y-axis for the large specimen: R- and T-directions, y-axis for the small specimen: T-direction), and then the bending vibration was generated by tapping the RT-plane of the bar at one end using a wooden hammer (tapping 1), while bar motion was detected by a microphone (Precision Sound Level Meter 2003, NODE Co., Ltd, Tokyo, Japan) at the other end (microphone placements 1 and 2). The signal was processed through the fast Fourier transform (FFT) digital signal analyzer (CF-5220, Ono Sokki Co., Ltd, Yokohama, Japan) to yield high-resolution resonance frequencies.

An iron plate with dimensions of 70.1 mm × 70.1 mm × 9.2 mm (approximately 347.4 g) for the large specimen and that with dimensions of 2 mm × 3 mm × 25 mm (1.30 g) for the small specimen were used as the concentrated mass and were bonded at $x = 0$ on the LR- and LT-planes of the large specimen and on the LR-plane of the small specimen with two-sided adhesive tape. The concentrated mass placements were 1 and 2 for the large specimen. A couple of different placement of iron plates was conducted because the center of gravity of the iron plate was changed by their placement. The bonding surface of the iron plate to the small specimen was a 2 mm × 30 mm plane. The values of μ were 0.0481–0.0569 for the large specimen and 0.0267–0.0578 for the small specimen, which were calculated using the masses of the concentrated mass and the specimen.

Table 1 Steps of vibration tests

Step	Dimension	Weight		Vibration test		
		Specimen	*M*	Free-free bending	Tapping RT-plane	
				Without *M*	Without *M*	With *M*
1	○	○				
2				○		
3					○	
4		○	○			
5						○
6		○				
7					○	
8				○		
9		○				

M concentrated mass, *RT* radial tangential

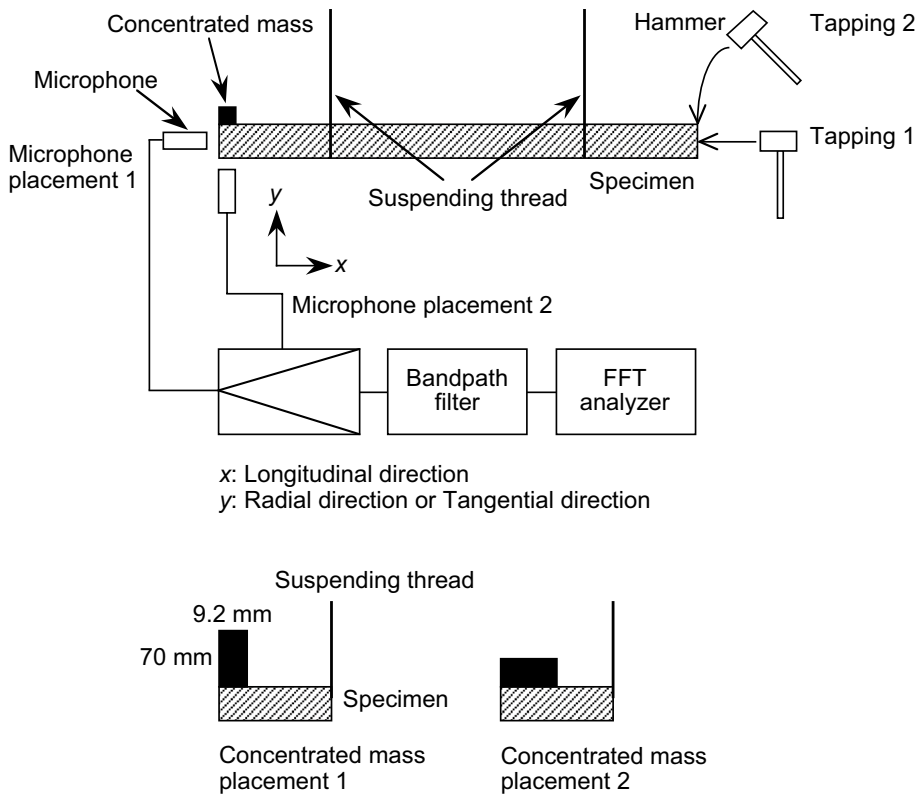

Fig. 2 A schematic diagram of the experimental setup for the free-free vibration tests

Bending vibration test

To compare the resonance frequency of bending vibration and that generated by tapping a cross section mentioned above, free-free bending vibration tests were conducted on the large and small specimens with and without the concentrated mass using the following procedure as shown in Fig. 2. The test bar was suspended by two threads at the nodal positions of free-free vibration corresponding to its first resonance mode, and then the bending vibration was generated by tapping the LR-plane (*y*-axis, T-direction) and LT-plane (*y*-axis, R-directions) of the large specimen and the LR-plane of the small specimen (*y*-axis, T-direction) at one end using the wooden hammer (tapping 2), while bar motion was detected by a microphone at the other end (microphone placement 2). The signal was processed through the FFT digital signal analyzer to yield high-resolution resonance frequencies.

Results and discussion

The means (standard deviations) of the density obtained using the weight and volume of the specimen and the Young's modulus using free-free longitudinal vibration without the concentrated mass were 405 (0.015) kg/m^3 and 10.94 (0.88) GPa for the large specimen and 533 (0.020) kg/m^3 and 16.62 (0.94) GPa for the small specimen, respectively, when the specimen weight became constant.

Figure 3 shows an example of waveforms generated by tapping the RT-plane of the large specimens (microphone placement 1 in Fig. 2). Various vibration modes appeared. The Young's modulus obtained by the Goens-Hearmon regression method based on the Timoshenko theory of bending [13–15] using the resonance frequencies of B1–B4 was 12.17 GPa while all the Young's moduli obtained by the longitudinal vibration theory using L1–L5 were 12.28 GPa, respectively. Because they were similar values, B1–B4 were the bending vibration modes and L1–L5 were the

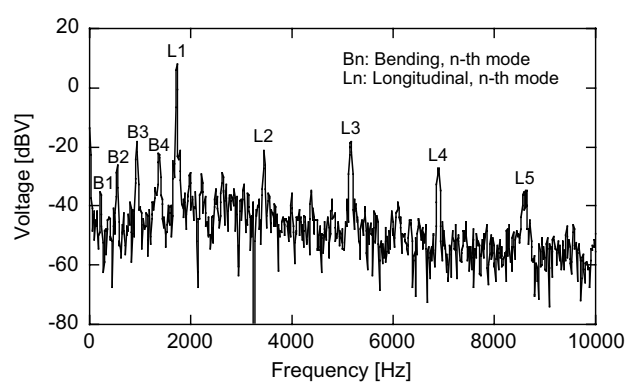

Fig. 3 An example of waveforms of large specimens

longitudinal vibration modes. Thus, the bending modes could be distinguished from the longitudinal modes.

The resonance frequencies without the concentrated mass generated by tapping the RT-plane (f_{10RT}) of the large specimens were similar to those generated by the free-free bending vibration (f_{10FF}) as shown in Table 2. The average (standard deviation) of the ratio f_{10RT}/f_{10FF} for the small specimens for 0 h–8 day (step 3/step 2 and step 7/step 8) was 1.0000 (0.00085). These results show that the appropriate resonance frequency of the bending vibration could be obtained by tapping the RT-plane.

The ratios of weight, density, and Young's modulus, estimated by the vibration method with additional mass, to those obtained by the normal method without the concentrated mass (W/W_0, ρ/ρ_0, E/E_0) were examined. The weight, density, and Young's modulus by the normal method (W_0, ρ_0, and E_0) were obtained as follows. The weight W_0 was the actual measurement. The density ρ_0 was calculated from W_0 and volume of the specimen. The Young's modulus E_0 was calculated using Eqs. (1) and (5). Resonance frequency was measured without the concentrated mass. These three ratios all show the same values ($W/W_0=\rho/\rho_0=E/E_0$) and indicate the estimation accuracy of the vibration method with additional mass. $W/W_0=\rho/\rho_0=E/E_0=1$ means that the estimation of weight, density, and Young's modulus by the vibration method with additional mass is perfect.

Table 3 shows the results of large specimens. Considering the ratio for uncorrected calculation ($a=0$ was used for estimation, "uncorrected" in Table 3), the estimation accuracy was slightly lower than the previous results using the similar size specimens [8]. When $a=70.1/2l=35.05/l$ and $a=9.2/2l=4.6/l$ were used for the calculation based on the center of gravity of the iron plate, the estimation accuracy was improved ("corrected" in Table 3).

Table 2 Comparison of the resonance frequency of the first mode without the concentrated mass generated by tapping the RT-plane (f_{10RT}) with that by the free-free bending vibration (f_{10FF})

Specimen	*y*-axis	f_{10RT} (Hz)	f_{10FF} (Hz)	f_{10RT}/f_{10FF}
1	T-direction	253.7500	253.7500	1.0000
	R-direction	255.0000	255.0000	1.0000
2	T-direction	252.5938	252.8125	0.9991
	R-direction	245.5938	245.6250	0.9999
3	T-direction	220.6563	220.7500	0.9996
	R-direction	222.7200	222.8125	0.9996
4	T-direction	231.5750	231.7500	0.9992
	R-direction	236.0250	236.0000	1.0001
5	T-direction	218.6500	218.6000	1.0002
	R-direction	220.0000	220.0000	1.0000

y-axis: refer to Fig. 2, f_{10RT}: Tapping 1 and Microphone placement 2 in Fig. 2, f_{10FF}: Tapping 2 and Microphone placement 2 in Fig. 2

The difference of resonance frequency between "*y*-axis: T-direction" and "*y*-axis: R-direction" for the same concentrated mass placement in Fig. 2 was very small such as 1–7 Hz. In this case, it is possible for a peak to split into two as shown in Fig. 4 because the shape of the cross section of the specimen is similar to a square and the difference of the shear modulus of the LR-plane and that of the LT-plane is not large [11]. When two peaks appear, it is difficult to choose the correct peak from the two peaks.

Consider the case when two peaks appear. Let f_{n0L} be the larger resonance frequency without the concentrated mass and f_{n0S} be the smaller resonance frequency without the concentrated mass. Let f_{nL} be the larger resonance frequency with the concentrated mass and f_{nS} be the smaller resonance frequency with the concentrated mass. In this case, substituting f_{nL}/f_{n0L} and f_{nS}/f_{n0S} into Eq. (6) are suitable. For the large specimen no. 1, using the pair of larger values ($f_{10}=255.0000$ Hz and $f_1=237.1875$ Hz, *y*-axis: R-direction, concentrated mass placement 1 in Fig. 2) and the smaller values ($f_{10}=253.7500$ Hz and $f_1=233.5313$ Hz, *y*-axis: T-direction, concentrated mass placement 1 in Fig. 2) is effective.

Figure 5 shows all results of the changes in the ratio of $W/W_0=\rho/\rho_0=E/E_0$ for the small specimens during drying at 20 °C and 65% RH. Roughly speaking, the ratio decreased and approached 1 as the moisture content decreased. When the resonance frequencies without the concentrated mass were values obtained from the first vibration test (step 2), the average (standard deviation) of the ratio was 1.08 (0.058), and when those were values obtained from the second vibration test (step 8), the average (standard deviation) of the ratio was 1.06 (0.050). The improvement of the estimation accuracy by using the second bending vibration test without the concentrated mass was similar to the previous study [8]. Since the moisture content of the small specimens continued to decrease during drying, the resonance frequencies without the concentrated mass obtained from the second vibration test was larger than those obtained from the first vibration test. Consequently, f_n/f_{n0} and m_n [Eq. (6)] obtained from the second vibration test were smaller than those obtained from the first vibration test. Since μ increases with the decrease in m_n [9], μ obtained from the second vibration test was larger than that obtained from the first vibration test. Since μ is expressed by Eq. (3), the weight obtained from the second vibration test was smaller than that obtained from the first vibration test. Therefore, one can conclude that generating bending vibration by tapping the RT-plane is effective for a drying process.

Table 3 Estimations by the vibration method with additional mass for the large specimens

Specimen	Without M				With M (vibration method with additional mass)								
	W_0 (kg)	ρ_0 (kg/m^3)	f_{10}(Hz)	E_0 (GPa)	f_1 (Hz)	Uncorrected				Corrected			
						W (kg)	ρ (kg/m^3)	E (GPa)	Ratio	W (kg)	ρ (kg/m^3)	E (GPa)	Ratio
Microphone placement 2, y-axis: T-direction, concentrated mass placement 1, tapping 1, refer to Fig. 2													
No.1	7.132	431	253.7500	12.04	233.5313	6.546	395	11.05	0.92	6.326	382	10.68	0.89
No.2	6.592	399	252.5938	11.10	229.9688	5.580	338	9.40	0.85	5.387	326	9.07	0.82
No.3	7.216	409	220.6563	11.17	205.5313	7.966	451	12.34	1.10	7.721	437	11.96	1.07
No.4	6.102	383	231.5750	9.48	211.7750	5.950	374	9.24	0.98	5.747	361	8.93	0.94
No.5	6.987	403	218.6500	11.04	202.7500	7.383	426	11.67	1.06	7.154	413	11.31	1.02
Microphone placement 2, y-axis: T-direction, concentrated mass placement 2, tapping 1, refer to Fig. 2													
No.1	7.132	431	253.7500	12.04	236.8750	8.280	500	13.97	1.16	6.341	383	10.70	0.89
No.2	6.592	399	252.5938	11.10	233.2188	6.887	417	11.60	1.04	5.235	317	8.82	0.79
No.3	7.216	409	220.6563	11.17	207.4375	9.433	534	14.61	1.31	7.384	418	11.43	1.02
No.4	6.102	383	231.5750	9.48	214.4500	7.225	454	11.22	1.18	5.503	346	8.55	0.90
No.5	6.987	403	218.6500	11.04	204.5000	8.569	495	13.55	1.23	6.688	386	10.57	0.96
Microphone placement 2, y-axis: R-direction, concentrated mass placement 1, tapping 1, refer to Fig. 2													
No.1	7.132	431	255.0000	12.17	237.1875	7.776	470	13.27	1.09	7.521	454	12.83	1.05
No.2	6.592	399	245.5938	10.45	224.1563	5.783	350	9.17	0.88	5.585	338	8.85	0.85
No.3	7.216	409	222.7200	11.38	207.2400	7.826	443	12.35	1.08	7.585	430	11.97	1.05
No.4	6.102	383	236.0250	9.44	217.5000	6.679	420	10.33	1.09	6.455	406	9.99	1.06
No.5	6.987	403	220.0000	11.15	203.7500	7.234	418	11.55	1.04	7.010	405	11.19	1.00
Microphone placement 2, y-axis: R-direction, concentrated mass placement 2, tapping 1, refer to Fig. 2													
No.1	7.132	431	255.0000	12.17	238.1250	8.332	503	14.21	1.17	6.382	386	10.89	0.89
No.2	6.592	399	245.5938	10.45	228.0313	7.547	457	11.96	1.14	5.758	349	9.13	0.87
No.3	7.216	409	222.7200	11.38	208.8000	8.951	507	14.12	1.24	6.995	396	11.03	0.97
No.4	6.102	383	236.0250	9.44	219.1750	7.563	475	11.70	1.24	5.771	363	8.93	0.95
No.5	6.987	403	220.0000	11.15	205.7500	8.560	494	13.66	1.23	6.679	386	10.66	0.96
Average (standard deviation) of 20 data									1.10 (0.12)				0.95 (0.081)

M concentrated mass, W weight, ρ density, f_1 resonance frequency of the first mode, E Young's modulus, Suffix 0: data without the concentrated mass, Ratio: $W/W_0 = \rho/\rho_0 = E/E_0$

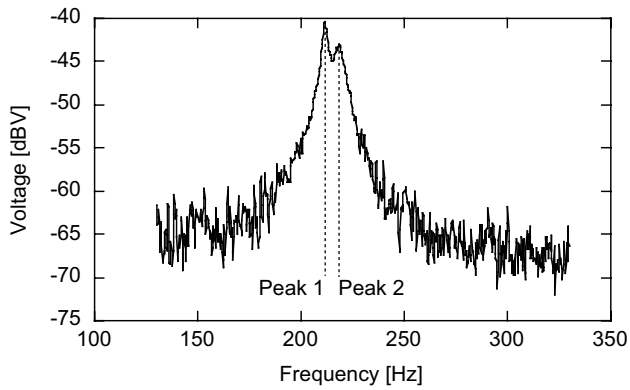

Fig. 4 An example of split peaks of large specimens

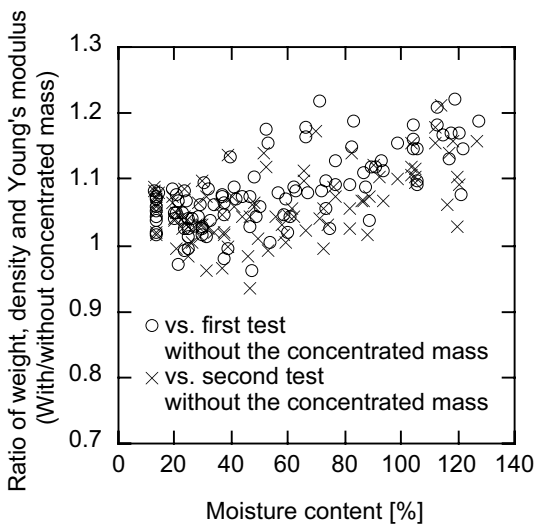

Fig. 5 Ratios of weight, density, and Young's modulus estimated by the vibration method with additional mass to those obtained by the normal method without the concentrated mass during drying at 20 °C and 65% RH

Conclusions

The resonance frequencies with and without the concentrated mass generated by tapping the RT-plane under the free-free condition were compared with those generated by the normal free-free bending vibration. Then, the weight, density, and Young's modulus estimated by the vibration method with additional mass using the bending vibration generated by tapping the RT-plane were examined. The following results were obtained:

1. The appropriate resonance frequency of the bending vibration could be obtained by tapping the RT-plane.
2. When two peaks appear, using the pair of larger values and the pair of lower values is effective.
3. Generating bending vibration by tapping the RT-plane was effective for the application of the vibration method with additional mass to a drying process.

Acknowledgements This study was supported by JSPS KAKENHI Grant number JP15K07522.

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